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It is hoped that this new venture will be welcomed by the research scholars and authors who are expected to come forward with their articles and research findings to nourish, sustain and promote this journal, which is a fresh leaf on the tree of knowledge.

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R. C. SHARMA
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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol 1 (1987), 1-6

EFFECTS OF *ALSTONIA SCHOLARIS* R. BR. EXTRACTS ON SEED GERMINATION AND EARLY SEEDLING GROWTH OF *PARTHENIUM HYSTEROPHORUS* LINN.

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ABSTRACT

The seeds of *Parthenium hysterophorus* Linn., when allowed to germinate in the different concentrations of aqueous extracts of fresh leaves, leaf litter and bark of *Alstonia scholaris* R. Br., the germination percentage, early seedling growth and emergence of first leaf were found to be inhibited and the ratio in the length of radicle and plumule was also disturbed.

Key words : Aqueous extracts, allelopathic potential, germination percentage, early seedling growth, concentrations, alkaloid.

INTRODUCTION

Since its inception in India, *Parthenium hysterophorus* Linn. (carrot weed) of family Asteraceae has become a troublesome weed for human beings. It has reduced the crop yield by 30–40% 1] and forage production by 90% [2]. It imposes allelopathic effects on many crops [3–6] and is responsible for dermal allergies in human beings and cattles [3, 7, 8]. At Rishikesh (30°, 07′ N and 78°, 19′E), it is spreading at a fast rate, to the exclusion of other species and has established itself as a dominant species. It's allelelopathic potential and high rate of reproduction of seeds appear to be the major causes of its dominance [6].

As the chemical methods, to control this weed are mostly hazardous, the biological control as the best long term solution, have been advocated by Hasler [9] and Sundara Rajulu and Gowri [10].

Alstonia scholaris R. Br. (Saptaparni) of family Apocyanaceae, si an evereen tree of Manu swamp and is dominating the area near Rishikesh. It is known for its economic and medicinal importance [11]. Based on the observations that under this tree carrot weed does not thrive well, the present study has been undertaken, to evaluate the allelopathic potential of *Alstonia scholaris* over *Parthenium hysterophorus*.

MATERIALS AND METHODS

The fresh leaves, leaf litter and bark of Saptaparni were oven dried at 80°C for 24 hr. 15 gm of separated, chopped parts were ground in 150 ml of distilled water separately. The solutions prepared were filtered and used as stock solutions or 100% concentration. The dilutions 50%, 25%, 10%, 5% and 2.5% were prepared by adding distilled water to stock solutions.

100 seeds of carrot weed were sown equidistantly in sterilized petridishes lined with double layer of filter paper. The solutions of various dilutionswere used as moistening agent using 15 ml per petridish. Distilled water served as control. After oue week of incubation at room temperature (30°±2°C) the germination percentage (taking the emergence of radicle as the criterion of seed germination), the length of radicle and plumule and emergence of first leaf were recorded, as has been recorded by Ghildiyal [12].

RESULTS

The fresh leaves, leaf litter and bark extracts of *Alstonia scholaris* showed the inhibitory action of various degree on seed germination and early seedling growth of *Parthenium hysterophorus* (Table 1)

In control the percentage germination observed was 96 and in lower concentration 2.5%, it was 92 in leaf extract, 88 in leaf litter extract and

84 in bark extract. The percentage germination decreased with the increase in the concentration of all extracts. The 100% concentration showed most inhibitory effect on germination.

The length of radicle and plumule was also inhibited in the extracts. As the concentration of extracts is increased the length of radicle and plumule decreases. The length of plumule was inhibited to a lesser extent in comparison to radicle, hence the ratio of radicle and plumule length was disturbed.

The number of seedlings in which the leaf appeared, was found to be reduced with the increase in the concentration of extracts. The number of seedlings showing the emergence of first leaf were reduced to zero percent, as have been observed in 50% and 100% concentrations of leaves and bark extracts.

DISCUSSION

The extracts of *Alstonis scholaris* imposed inhibitory effects on percentage germination, radicle-plumule growth and emergeence of first leaf of *Parthenium hysterophorus*, just like aqueous extracts of many plant species which inhibit the germination and seedling growth of several other plants [12-13]. The allelopathic potential of *Alstonia* over *Parthenium* may be correlated with the presence of alkaloids, mainly echitamine, which has been repo-ted as the chief alkaloid in *Alstonia* by Goodson in 1928 and Siddappa in 1945 [11]. Due to the presence of achitamine, it was previously used as a substitute of quinine [11]. The allelopathic potential of *Alstonia*, is supported with the fact that the bark extracts were found most inhibitory, as the bark of this plant contains the highest amount of alkaloids [11] The degree of inhibition is thus correlated with the amount of alkaloids present in the extracts.

In control, the length of radicle was more than the plumule, but their ratio is disturbed as the concentration increases. In higher concentrations of extracts the length of plumule was more than that of radicle. This is, due to the fact that the plumule was affected but to a lesser extent in comparison of radicle, in the presence of alkaloids.

Table 1 : Seed germination and early seedling growth of *Parthenium hysterophorus* in leaf, leaf-litter and bark extracts of **Alstonia scholaris** as observed after one week.

Plant part Extract	Concentration of Extracts	Germination (%)	Length (cm)* Radicle	Plumule	Emergence of First leaf (%)
Leaf Extracts	2. 5 %	92	1.43 ± 0.39	1.11 ± 0.27	56
	5. 0 %	84	1.42 ± 0.53	1 08 ± 0 38	48
	10. 0 %	60	1.40 ± 0.50	0.98 ± 0 25	32
	25. 0 %	25	1.00 ± 0.11	0.75 ± 0.2	18
	50. 0 %	20	0.96 ± 0 27	0.73 ± 0.15	—
	100. 0 %	12	0.63 ± 0.12	0.71 ± 0.17	—
	Control	96	2 83 ± 0.38	1.28 ± 0.47	80
Leaf-litter Extract	2.5 %	88	1.86 ± 0.39	1.20 ± 0.28	72
	5 0 %	84	1.48 ± 0 28	1.16 ± 0.25	68
	10 0 %	64	1.28 ± 0.28	1.09 ± 0.22	56
	25.0 %	48	1.08 ± 0 15	1.00 ± 0.15	40
	50.0 %	36	0 84 ± 0.09	0.86 ± 0.13	26
	100.0 %	24	0 40 ± 0 10	0.80 ± 0.12	20
	Control	96	2.36 ± 0.39	1.48 ± 0.27	80
Bark Extract	2.5 %	84	1.25 ± 0.33	1.66 ± 0.41	60
	5.0 %	80	1.20 ± 0.10	1.05 ± 0.25	40
	10.0 %	60	1.09 ± 0.27	0.92 ± 0.22	32
	25.0 %	48	1.00 ± 0.11	0.84 ± 0.14	20
	50.0 %	12	0.70 ± 0.15	0 82 ± 0.10	—
	100.0 %	08	0.53 ± 0.07	0.76 ± 0.08	—
	Control	96	2.34 ± 0.41	1.78 ± 0.39	84

* Mean ± Standard Deviation.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 1 (1987), 7-14

GROWTH RATE, PRODUCTION AND ENERGY ACCUMULATION EFFICIENCY OF *DENDROCALAMUS STRICTUS* NEES. (BAMBOO) IN WESTERN HIMALAYA, INDIA

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ABSTRACT

The present paper reports on the growth rate, biomass accumulation pattern and energy conserving efficiency (ECE) of a sub-tropical species of Bamboo (*Dendrocalamus strictus* Nees.) in western Himalayan forest. The maximum growth rate and ECE were recorded at VI stage of growth and amounted to 36.7 cm day^{-1} and 3.00% respectively. However, the peak value for biomass was observed at XI stage of growth (3299 g $culm^{-1}$). Plants attained the maximum of 1055.[illegible] cm in 55 days. Significant correlation existed between the length and the biomass of the culm.

Key words : Bamboo, Growth rate, Biomass, Net energy conversion rate, Energy conserving efficiency.

INTRODUCTION

Dendrocalamus strictus Nees. (bamboo) forms the middle storey throughout the sub-tropical mixed deciduous forest of western Himalaya (Champion and Seth [3]). It is a giant tree-like grass, and has been put to innumerable uses since time immemorial. The importance of bamboo in human life throughout the tropics and sub-tropics has been extolled by various artists and apicures. Studies on various parameters of bamboo (*D. strictus*) have been carried over by several workers (see for instance, Ingle and Bose [6], Riccharia and Kotwal [12], Singh [13] and Yadava *et al.* [14]). A few reports on ecological behaviour of the bamboo are also available (Joshi and Purohit [8], Agrawal and Joshi [1]). However, studies on the growth rate behaviour and biomass accumulation are lacking. In the present investigation, these parameters including the growth pattern of culms, production, standing crop of energy and energy conserving efficiency of *D. strictus* were studied during 1984-85.

MATERIALS AND METHODS

The study area : The study area is located at Lat. 29° 46′ N and Long. 78° 30′ E, in the western Himalaya. The mean elevation above mean sea level ranges between 395 - 450 m. Climatically, the zone is sub-tropical (Champion and Seth [3]). The mean annual rainfall is 1263 mm, 75% of which occurs in rainy season (July-August). From October onwards, occasional showers occur, Durin November - February, the weather is usually uncertain when winter showers with severe cold winds are marked features. Sandstones are the characteristic rock type (Audin [2]). The slopes are moderate with irregular topography.The soil is sandy loam and the clay percentage increased with the depth. The pH of the soil ranges between 7.9 to 8.9. The soil is poor in organic carbon, total nitrogen and phosphorus contents. The forest vegetation comprises ***Shorea robusta, Buchnania langen, Anogeissus latifolia, Albizia lebbeck*** and ***Emblica officinalis. Dendrocalamus strictus*** forms the middle storey in the forest.

Sampling : Two thousand culms of bamboo in different clumps with culm density in the initial stage of growth were selected for the present study after the break of rains in August 1984. The sampling was made at five days intervals and growth during this period (5 days) was considered as stage of growth. At each stage of growth, 50 culms were removed and measured for height. For biomass study, culms were dried at 80° C in hot air oven and weighed. The values are expressed as g culm^{-1}. Regression equations were developed between different growth attributes Energy values were taken from Golley's conv. rsion table [5]. The standing crop of energy was calculated by multiplying the energy values with standing crop of different stages of growth. NECR (Net energy conversion rate) and ECE (Energy conserving efficiency) were calculated following Lieth [10] :

$$\text{NECR} = \frac{\text{kcal}_2 - \text{kcal}_1}{t_2 - t_1}$$

where kcal_1 and kcal_2 are the standing states of energy per unit ground area at the time t_1 and t_2 respectively.

$$\text{ECE}(\%) = \frac{\text{Energy captured by the net dry matter production} \times t \times 100}{1/2 \text{ solar radiation} \times t}$$

where t = Period of solar radiation and energy captured.

RESULTS AND DISCUSSION

Growth of culms were recorded for two months divided into 11 stages, each stage being represented by 5 days of growth. Table 1 shows the growth rate of *D. strictus.* The maximum growth was recorded at stage VI (36.7 cm day $^{-1}$) and minimum at stage II (2 0 cm day $^{-1}$). The growth declined from stage VI onwards and at the last stage it was 21.0 cm day $^{-1}$. The total length of the culm after each successive interval (5 days) of the study has also been incorporated in Table 1. The initial length of the culm was 14 cm which reached 1055.0 cm at the last stage of growth. An abrupt increase in the length was found after stage III.

The results of biomass and productivity at different stages have been given in Table 2. The initial biomass, 44 0 g culm $^{-1}$, inereased till the maximum was recorded at XI stage (3299 g culm $^{-1}$). The maximum and minimum production was at stage VI (574.00 g culm^{-1}) and II (31.00 g culm $^{-1}$) in five days. The diameter of the culm was fixed at VI to VII stage, onwards.

Table 3 shows the standing crop of energy, net energy conserving rate and energy conserving efficiency. The maximum standing crop of energy was found at stage XI (6087 kcal m $^{-2}$) and minimum at stage I (81 kcal m $^{-2}$). Maximum net energy conserving rate was found at stage VI (212 kcal m $^{-2}$ day $^{-1}$) and minimum at stage II (12 kcalm $^{-2}$ day $^{-1}$). Likewise, the maximum energy conserving efficiency (%) was observed at stage VI (3 0%) and minimum at stage II (0.20%).

Statistical relationship between length and biomass was calculated. The regression equation between the two parameters is as follows:

$$Y = 5.5326 X + 14.132$$

where X = Length in cm and Y = Biomass in g culm $^{-1}$.

Correlation coefficient was significant at 5% level (r = + 0.94).

The vegetative reproduction of this crop occurs mainly through culm production. In India, tropical species, *Bambusa tulda*, is on record, having rapid growth rate of 3.6 cm in one hour, and so in North Africa Phylostachys mitis, a hardy species is known to have achieved the phenomena growth rate of 48 cm day^{-1} [9]. In the present investigation, *D. strictus* has been found growing at the rate of 19.2 cm day^{-1}.

The highest percentage of ECE in *D. strictus*, as recorded in this study (3.0%), compares with that of *Poa compressa* (1.1%) and old field sedge in North America (3.9%) as reported by Ovington and Lawrence [11].

TABLE 1. Growth rate of bamboo (*D. strictus*) in western Himalaya

Stages of growth	Growth rate (cm day^{-1})	Total length after 5 days interval (cms)
I	–	14 ± 7.42
II	2 ± 1.76	24 ± 11.23
III	4 ± 2.28	44 ± 16 54
IV	21.8 ± 9.62	153 ± 34.55
V	29.0 ± 11.76	298 ± 56.55
VI	36.7 ± 19.90	481.7 ± 65.55
VII	22.0 ± 12.21	598 ± 104.32
VIII	28 0 ± 9.65	730 ± 189.78
IX	22.7 ± 8.89	883 ± 156.85
X	21.4 ± 10.24	950 ± 167.45
XI	21.0 ± 9 55	1055 ± 188.87

The values are significantly different at 5% level in t test

TABLE 2. Biomass (g culm^{-1}) and product on (g per 5 days) for various stages of growth of *D strictus* in western Himalaya

Stages of growth	Biomass g culm^{-1})	Production (g per 5 days)
I	44.0 ± 12.27	44.00
II	75.0 ± 24.43	31.00
III	138.0 ± 39.45	63.00
IV	478.0 ± 125 55	341.00
V	932.0 ± 223.75	453 00
VI	1505.0 ± 325.24	575.00
VII	1850.0 ± 345.56	344.00
VIII	2284.0 ± 426.73	433 00
IX	2639.0 ± 343.55	354.00
X	2971.0 ± 455.56	332 00
XI	3299 0 ± 478 76	328.00

The values of biomass are significantly different in t test.

TABLE 3. Standing crop of energy (kcal m^{-2}), net energy conserving rate (NECR) (kcal $m^{-2}day^{-1}$) and energy conserving efficiency (ECE %) at different stages of growth in *D. srictus* in western Himalaya

Stages of growth	Standing crop of energy (kcal m^{-2})	NECR (kcal m^{-2} day^{-1})	ECE (%)
I	81.00	16.0	0.28
II	138.00	12.0	0.20
III	254.00	23.0	0.32
IV	882.00	126.0	1.80
V	1719.00	167 0	2.30
VI	2779.00	212.0	3.00
VII	3414.00	127.0	1 80
VIII	4213.00	160.0	2.20
IX	4868.00	131.0	1.80
X	5481.00	123.0	1.70
XI	6087.00	121.0	1.70

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol 1 (1987) 15–

CHEMICAL INVESTIGATION OF SOME INDIGENOUS BASIL OILS

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ABSTRACT

Ocimum basilicum L., distributed in and around Rishikesh comprises of a mixture of two chemo-types,viz.,'methyl chavicol type'and'eugenol type'. Seeds of these plants when raised at Haldwani, another place on foothills of Himalayas gave a true 'French basil'(rich in methyl chavicol & linalool). Rishikesh oil sample contained greater percentage of alcohols, lesser percentage of phenols/phenolic derivatives and hydrocarbons than Haldwani sample. GLC study of the oils revealed the presence of twenty one compounds in Rishikesh sample and fifteen compounds in Haldwani sample.

Key words and phrases : Essential oil, Basil oil.

Chemistry-subject classification : Pharmaceutical Analysis — Essential oils-62.

Botanical classification of Basil (*Ocimum basilicum* Linn.) is complicated due to existence of many varieties and forms. Several designations have often been assigned to one and the same type; polymorphism and cross pollination under cultivation have often given rise to a number of sub-species, varieties and races. Owing to this difficulty users have had to resort to classify the essential oil from *O. basilicum* Linn. according to their chemical composition and their production areas. Guenther has grouped these oils into four Basil varieties [2] [3] ;

1. European type : main constituents- linalool and methyl chavicol.
2. Re:union type : main constituent- methyl chavicol.
3. Methyl cinnamate type : main constituent- methyl cinnamate.
4 Eugenol type : main constituent- eugenol.

In the present course of work the oil from *O. basilicum*, a widely and wildly distributed plant in and around Rishikesh, has been studied in order to know the exact chemical composition of the oil and to which chemical type it belongs. Another oil sample extracted from the plants raised at Haldwani (anothor place on foothills of Himalayas) from the seeds of *O. basilicum* collected at Rishikesh, has also been studied for its chemical composition to know the change in chemical composition of the oil, if any, due to climate and related factors.

Experimental : The essential oil was obtained by steam distilling the plant in a copper still, extracting the distillate with petroleum (b. 60-80°C), drying (sodium sulphate) and concentrating the organic phase under reduced pressure (yie!d - 0.39 percent). Physico-chemica l constants of the oil were :

Sp. gravity/20°C=0.9160; Ref. Index/20°C= 1.473; Op. rotation/20°C =-4.0°; Acid value=2.2; Ester value = 15.2; Solubility in 80%

Alcohol/20°C = 1 : 0.9; Ester content as Eugenol acetate = 5.82; and Eugenol content = 15%.

Presence of acetic acid and n-butyric acid was detected in essential oil by circular paper chromatography and TLC [5], [7] and [1].

Analysis of the acid free oil involving chemical separation, derivatisation and TLC techniques [4],[8]and[9] showed the presence of thirteen constituents, viz., eugenol, iso-eugenol, terpinolene, humulene, cadinene, caryophyllene, linalool, borneol, 1 : 8 cineole, caryophyllene oxide, methyl iso-eugenol, methyl eugenol, eugenol acetate and methyl chavicol.

Chemical composition of the oil was also studied by GLC using Perkin Elmer model 3920 under the following conditions :

Column : 3.6m long; 2.0mm I.D.
Stationary phase : UCCW 10% in chromosorb, 80-100 mesh.
Column temperature : 70-200°C; @ 2°/minute.
Carrier gas : Nitrogen @ 35 ml/minute.
Chart speed : 5mm/minute.
Detector and
Injector temperature : 250°C
Attenuation : 512 and 256,
Range : 10
Sample size : 4 μl–6 μl (1 : 30 solution in chloroform).

The GLC of the oil sample showed twenty seven peaks. The compounds in the gas chromatogram were identified by comparing their retention time with those of standard compounds. These results were also confirmed by the "enriching method", carried out earlier by Lawrence [8] on Comoro basil oil. The results of the GLC study are mentioned in Table 1.

Chemical composition of essential oil from the plant *O. basilicum* raised at Haldwani (Nainital), from the seeds of the plants found growing wild at Rishikesh, was also examined by GLC, under above conditions using Perkin Elmer model 3920, in order to know the difference in chemical composition due to climate and other factors. Results of GLC study are mentioned in Table 1.

Results and discussion : The essential oil extracted from the plants distributed at Rishikesh possessed the physico-chemical properties as mentioned in experimental part of the present investigation. Besides the persence of acetic acid and n-butyric acid the oil was found to contain 27 peaks of which 21 were identified (cf. Table 1).

The essential oil, extracted from the plants raised at Haldwani from the seeds of plant *O. basilicum* L. distributed at Rishikesh, was found to have : Sp. gravity/20°=0.9424; Ref. index/20°=1.503; Op. rotation/20°= –3° 20°; Acid value=0.88; Ester value= 10 30; Carbonyl value= 1.4 (as $C_{10}H_{16}O$). The oil was found to contain twenty peaks of which fifteen were identified (cf. Table 1).

Study of the two oil samples belonging to the same *O. basilicum* variety collected and raised at different places revealed that the Rishikesh sample contained greater percentage of alcohols (54.05%), and lesser percentage of phenols/phenolic derivatives (34.4%) and hydrocarbons (5.4%) than the Haldwani sample which contained alcohols (45.45%), hydrocarbons (6.78%) and phenols/phenolic derivatives (40 34%).

Review of literature reveals that the basil oil containing appreciable amount of linalool (more than 40%), and methyl chavicol (more than 30%) is classified as the true 'French basil oil' [6], [10]–[11]. Rishikesh variety of *O. basilicum* can not be classified as true 'French basil' while the plants raised at Haldwani from its seeds yielded the oil, containing 45.2% linalool and 32.02% methyl chavicol, fulfills the requirements of 'French basil oil'. It appears that the plants distributed at Rishikesh is a mixture of two races, viz., 'methyl chavicol type' and 'eugenol type'. Further work is needed to ascertain the factors responsible for this transformation.

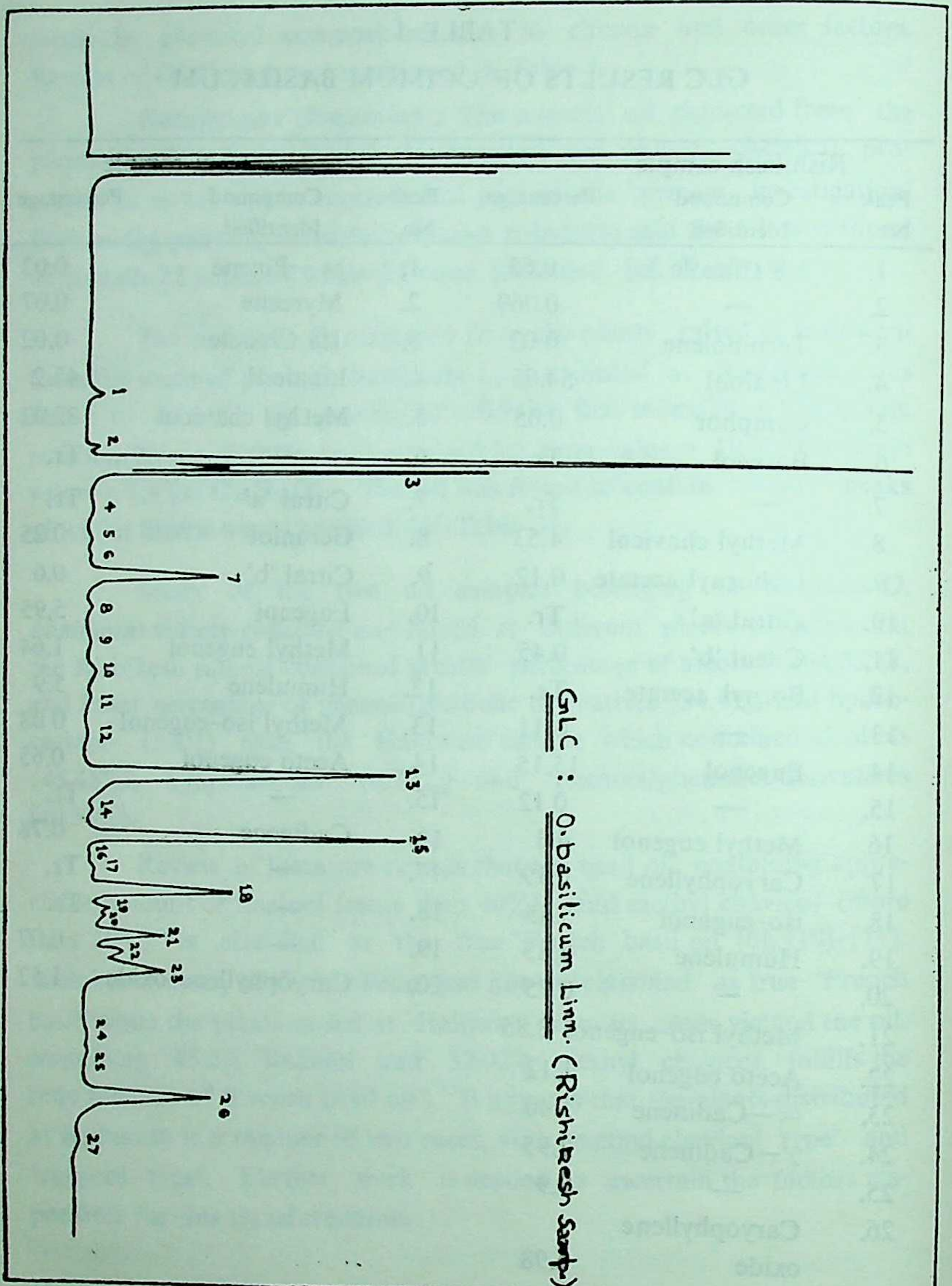
It is for the first time that the naturally distributed plants of *O. basilicum* are being reported producing true 'French basil' oil.

TABLE 1

GLC RESULTS OF OCIMUM BASILICUM

Rishikesh sample			Haldwani sample		
Peak No.	Compound Identified	Percentage	Peak No.	Compound Identified	Percentage
1.	1:8 Cineole	0.65	1.	∝—Pinene	0.03
2.	—	0.069	2.	Myrcene	0.07
3.	Terpinolene	0.02	3.	1:8 Cineole	0.02
4.	Linalool	54.05	4.	Linalool	45.2
5.	Camphor	0.05	5.	Methyl chavicol	32.02
6.	Borneol	Tr.	6.	—	Tr.
7.	—	Tr.	7.	Citral 'a'	Tr.
8.	Methyl chavicol	4.53	8.	Geraniol	0.25
9.	Isobornyl acetate	0.12	9.	Citral 'b'	0.6
10.	Citral 'a'	Tr.	10.	Eugenol	5.95
11.	Citral 'b'	0.45	11.	Methyl eugenol	1.64
12.	Bornyl acetate	Tr.	12.	Humulene	5.9
13.	—	0.11	13.	Methyl iso-eugenol	0.08
14.	Eugenol	15.15	14.	Aceto eugenol	0.65
15.	—	0.12	15.	—	Tr.
16.	Methyl eugenol	4.3	16.	Cadinene	0.78
17.	Caryophyllene	0.19	17.	—	Tr.
18.	iso-eugenol	5.45	18.	—	Tr.
19.	Humulene	0.85	19.	—	0.07
20.	—	0.75	20.	Caryophyllene oxide	1.52
21.	Methyl iso-eugenol	3.85			
22.	Aceto eugenol	1.12			
23.	∝—Cadinene	1.40			
24.	γ—Cadinene	2.95			
25.	—	0.9			
26.	Caryophyllene oxide	5.98			
27.	Farnesol	Tr.			

In this table – denotes unidentified peaks.



GLC : O. basilicum Linn. (Rishikesh Samp.)

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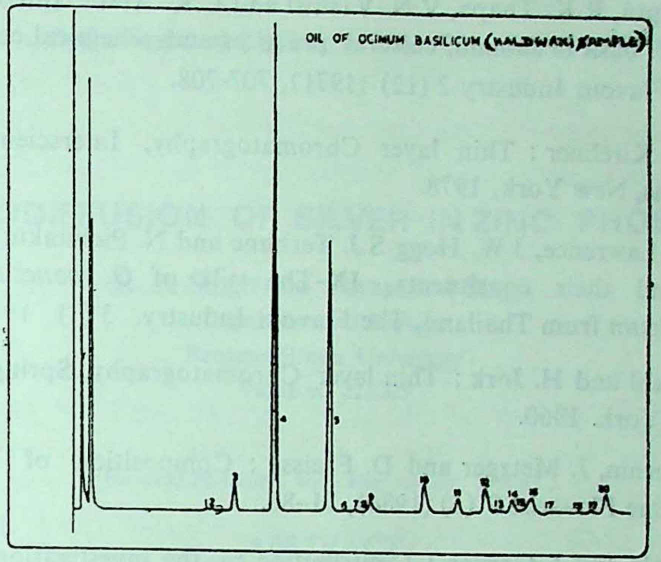

GLC- O basilicum (Haldwani Sample)

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 1 (1987), 23-30

ELECTRODIFFUSION OF SILVER IN ZINC PHOSPHIDE

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ABSTRACT

Electrodiffusion of silver into Zn_3P_2 films is studied. Experiment was perfomed using $Ag/n\text{-}Zn_3P_2/Ag$ flat geometry. The diffusion coefficient was found to be a function of applied eleciric field. It has been observed that the electric field assisted diffusion starts at relatively higher electric field ($>6.4\times10^3$ V/cm). Electrodiffusion of silver in the $n\text{-}Zn_3P_2$ films results in the formation of relatively high resistive film material. The electrical resistivity of the formed film depends critically on the magnitude and duration of the applied electric field during the formation of the sample (electrodiffusion). The rate of diffusion increases with the applied electric field. The reuslts are axplained assuming that the electrodiffused silver forms acceptor states in the film material.

Kewords and phrases : Electrodiffusion, semiconductor, resistivity.
Subject classification : Physics.

INTRODUCTION

Zinc phosphide is a promising material for solar cell applications because of its optimum band gep (~ 1.5 eV) and large value of minority carrier diffusion length. A number of Zn_3P_2 based Schottky barrier and heterojunction solar cells have been fabricated and tested during the past few years. An attainable efficiency of 18.5% has been predicted for the Mg/Zn_3P_2 metal-semiconductor Schottky barrier photovoltaic solar cell whereas the efficiency achieved is ~ 8% for single crystal devices [2] and ~ 6% for polycrystalline thin film devices [1]. So far, the performance of these cells has been improved with an empirical approach. The maximum plausible value of open circuit voltage (V_{oc}) appears to be between 1/2 and 2/3 of E_g. The cell with improved fill factor may reach an efficieney of more than 12%, at least in the single crystal form. Such high performance will require, apart from advances in fabrication technology, device analysis and device theory, some understanding of the material properties of Zn_3P_2. This paper deals with electrodiffusion of silver in Zn_3P_2.

The impurities and structural imperfections cause energy states in the band gap of the semiconducting materials and strongly influence the current flow through such devices. For example, room temperature resistivities for single crystalline and polycrystalline samples of Zn_3P_2 have been reported in the range 10^2 tp 10^5 Ω -cm [1]. This wide range of resistivity has been ascribed to the variation in the density of trapping centres which depends upon the deposition conditions [3]. Wang *et al.* [6] have found a decrease in electrical resistivity of the p-Zn_3P_2 single crystals kept in contact with silver in vacuum at 573 K for 120 hours. This was explained in terms of thermal diffusion of silver acceptors into the material which was ascribed from the measurement of hole number density (silver ions) found to increase at the end of the experiment. The present paper deals with the I-V characteristics of vapour deposited n-Zn_3P_2 films at low and high fields. The marked deviation in the I-V characteristics at high fields is explained in terms of electrodiffusion of silver acceptors into the film material.

EXPERIMENTAL

Zinc phosphide films were obtained by evaporating pure Zn_3P_2 material (in the pellet form) from a clean molybdonum boat at 10^{-6} torr on thoroughly cleaned glass plates. Simultaneously four to five films of appropriate size were obtained with the help of suitable masks. Film thickness was measured in-situ with the Quartz Crystal Thickness Monitor (Model OFTM-1). Source to substrate distance was kept 15 cm. For electrical measurements, vacuum evaporated high purity grade silver films on glass plates were used as back contacts. The cell geometry used for I–V measurements was $Ag/Zn_3P_2/Ag$ as shown in fig. (1). Many silver top electrodes provided many sample cells for measurement. One cell was used at one time. The current was measured by varying the voltage.

Further, for a fixed field, the number of electromigrated silver atoms (ions) shall increase with time and hence current with time would change. The variation of current with time was also measured using $Ag/n\text{–}Zn_3P_2/Ag$ system of flat geometry shown in fig. (2). First, Zn_3P_2 film was deposited on a glass substrate. Later, two rectangular silver electrodes separated by a small distance, were deposited by vacuum evaporation. During electrode deposition, a thin wire (diameter $\sim$ 0.008 cm) was used as a mask. A 300 V RADART dc power supply was used as a source of high field. The voltage was recorded by a digital voltmeter and the ourrent with a digital microammeter.

RESULTS AND DISCUSSION

A typical log I vsrsus log V plot is shown in fig. (3). At low fields an ohmic behaviour is seen. The I–V characteristics in this region remained linear even on reversal of the polarity. Film resistivity ($\varrho = \frac{RA}{t}$), calculated by measuring the resistance R of the film, the area A of the electrode (silver dot) and film thickness ($t = 0.4\ \mu m$) is found to be $2.3 \times 10^3\ \Omega$- cm. At high fields, the I–V characteristics deviates from ohmic behaviour and film resistivity increases, first slowly and then abruptly beyond a field greater than 2.5×10^4 V/cm. The resistivity at 2×10^4 V/cm becomes $3.7 \times 10^3\ \Omega$- cm. The relative decreases in current at high fields amounting to an increase in the sample resistance deserves special attention. In p—type Zn_3P_2, a decrease in resistance was noted by Wang *et al.* [6] when metallic silver was diffused into the bulk at 573 K which was explained by invorking that the thermally diffusing silver forms "acceptor states". It was based on the measurement of hole density which was found to increase after silver diffusion. In our case, we have assumed an "electrodiffusion" for silver under high fields. Since our starting material is $n\text{–}Zn_3P_2$, introduction of additional acceptor states as a result of Ag—electrodiffusion would lead to an increase in the film resistivity. Higher is the electric field higher will be the electrodiffusion and subsequent increase in resistivity. Further, for a fixed field, the amount of electrodiffused silver shall increase with time and hence current will decrease.

Raleigh and Crowe [5] have shown that the diffusion current I decreases with time t and follows the well known Cottrell relation

$$I = FA\,(C_1 - C_0)\,(\frac{D}{TT})1/2\,t^{-1/2}$$

where A is the electrode area, D is diffusion coefficient, C_0 and C_1 are silver concentrations corresponding to voltage dependent chemical activities $_{e}{-V_0}\,F/RT$ and $_{e}{-V_1}\,F/RT$. Thus, if voltage is changed from V_0 to V_1 and current I is measured as a function of time t; then a plot of I versus $t^{-1}/^{2}$ will be a straight line whose slope will depend upon the applied voltage/electric field. The measurement was done using the geometry of fig. (2).

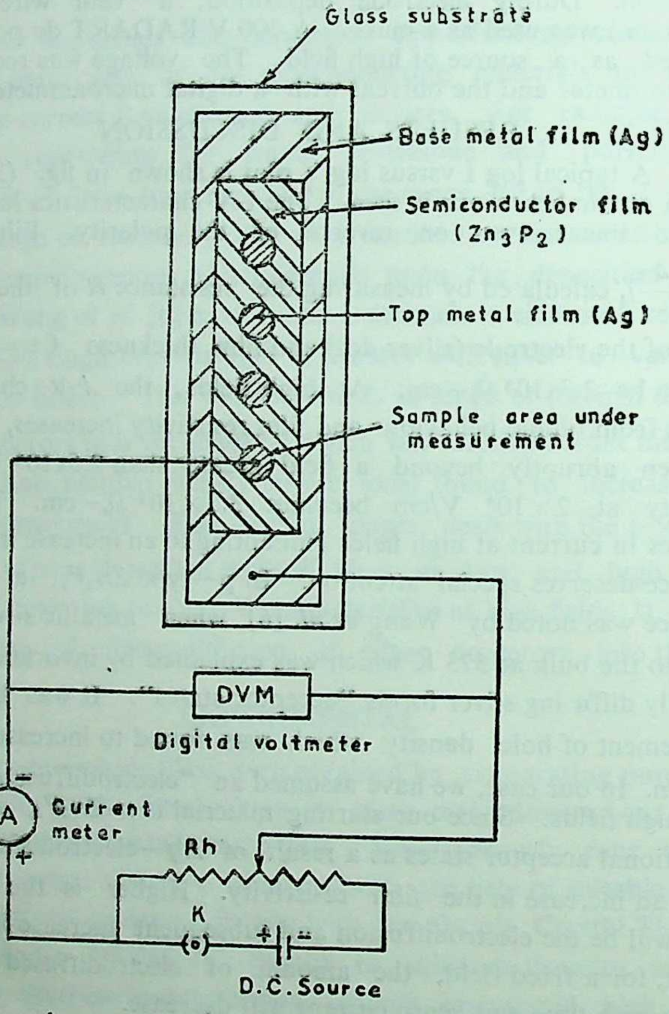
Glass substrate
Base metal film (Ag)
Semiconductor film
(Zn_3P_2)
Top metal film (Ag)
Sample area under measurement
DVM
Digital voltmeter
A
Current meter
Rh
K
(o)
D.C. Source

Fig. 1 : Sample geometry for current–voltage measurements.

The plots of I versus $t^{-1/2}$ for various applied electric fields are shown in fig. (4). It is observed that for $E < 6.20 \times 10^3$ V/cm, the current is almost constant with time indicating that there is negligible diffusion at or below this field (curves for $E < 6.3 \times 10^3$ V/cm are not shown in the figure), where as for $E \geqslant 6.4 \times 10^3$ V/cm, the current changes with time. The plots of I versus $t^{-1/2}$ are straight lines which confirm the diffusion [4] or electromigration of silver in Zn_3P_2. Also the slope of the curve increases with the applied electric fields indicating an increase in the diffusion rate with electric field as expected. The present experiment could not be performed at higher applied fields $E \geqslant 9 \times 10^3$ V/cm due to sparking in the air gap between silver electro-

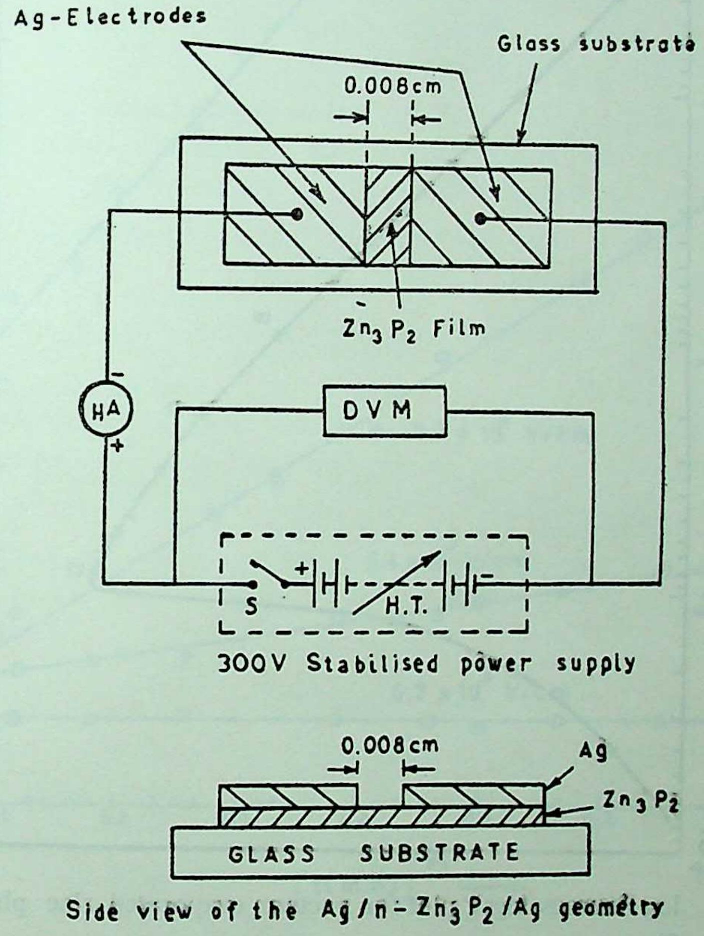
Ag-Electrodes
Glass substrate
0.008cm
Zn$_3$P$_2$ Film
−
μA
+
DVM
+
−
S
H.T.
300V Stabilised power supply
0.008cm
Ag
Zn$_3$P$_2$
GLASS SUBSTRATE
Side view of the Ag/n-Zn$_3$P$_2$/Ag geometry

Fig 2 : Flat geometry used in the study of Electrodiffusion experiment.

des. In addition to the above experiment, some measurements on Hall effect is planned from which information about hole density and hole mobility may be derived. These measurements would further shed light on the electrodiffusion of silver into the Zn_3P_2 material.

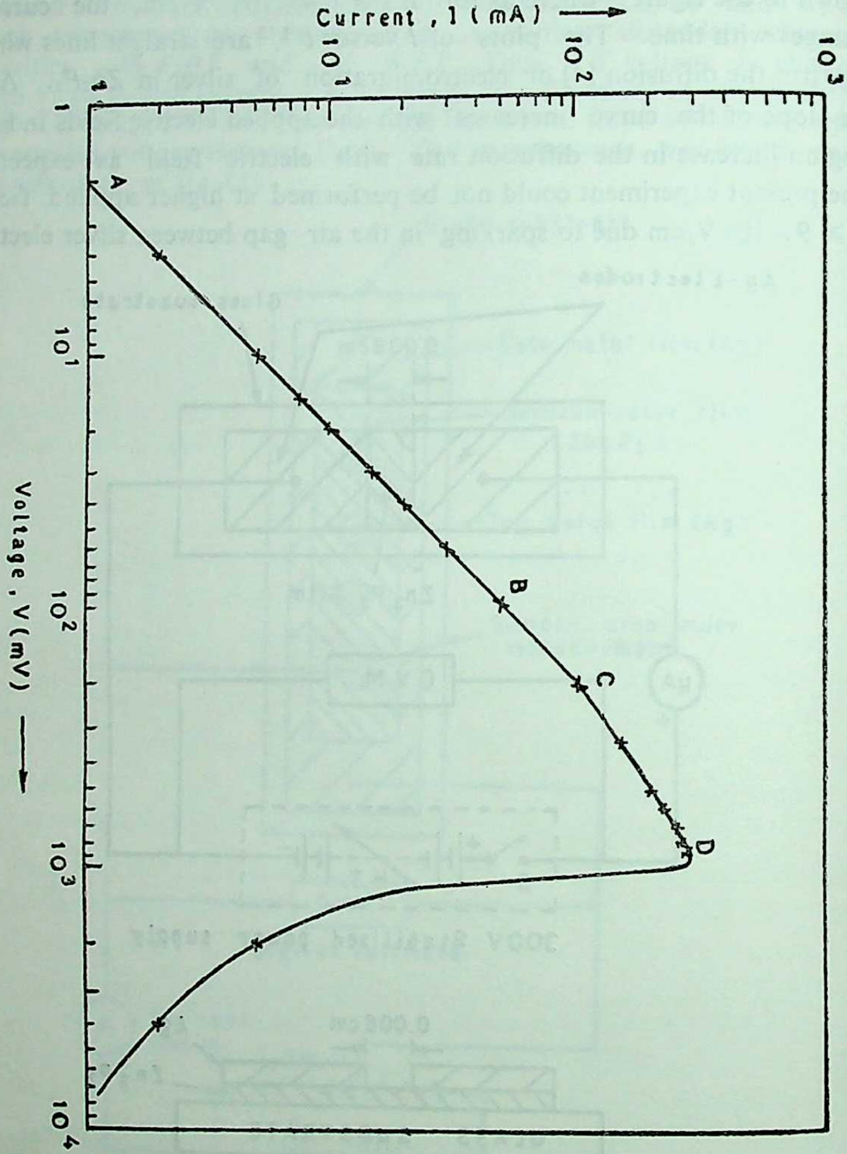

Fig. 3 : log *I* versus log *V* plot for vacuum evaporated zinc phosphide film.

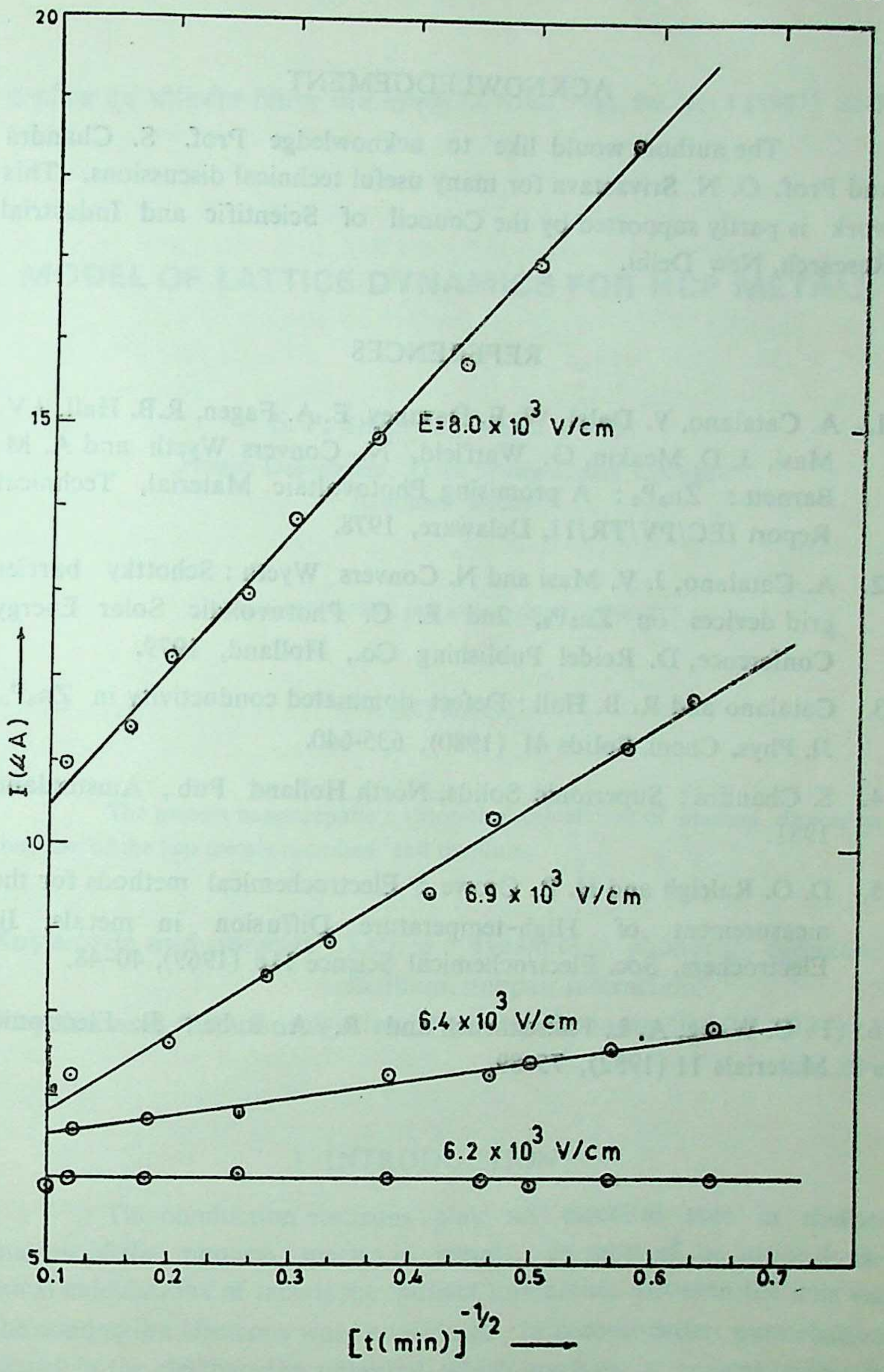

Fig. 4 : Plots of *I* versus $t^{-1/2}$ for various applied electric fields.

ACKNOWLEDGEMENT

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol 1 (1987) 31-38

MODEL OF LATTICE DYNAMICS FOR HCP METALS

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ABSTRACT

The present paper reports a theoretical calculation of phonon dispersion relation of the hcp metals cadmium and thallium.

Keywords and phrases : Lattice dynamics, phonons, cadmium, thallium, nonpair interaction.

Physics and Astronomy Classification Scheme (PACS 1977) : 63.20 e

I INTRODUCTION

The conduction electrons play an essential role in the formation of the phonon spectra in metals. In most of the lattice dynamical calculations of metals the indirect interaction between the ions via the conduction electrons was considered in second-order perturbation theory in the electron–ion potential, which involves a central pairwise interionic interaction and leads to real electronic contribution. The same is true for coulombic contribution. An important result was obtained in an electron ion plasma in a metal, where, in addition to the usual pairwise interaction, there exists a nonpair interaction via conduction electrons. Physically such an interaction is related to the scattering of electrons by three or more metallic ions simultaneously. As an extension of second-order perturbation theory, Brovman and Kagan [3] have shown that the correction in dynamical matrix arising from higher than second-order scattering of conduction electrons by usual pseudopotential will give rise to tensor forces Brovman et al. [4] have developed a method for separating long-range pairwise interaction and short-range nonpairwise interaction between the ions. They have suggested that the correct description of the whole phonon spectrum to the *s–p* bonded hexagonal metals can be obtained only when unpaired forces are included in the dynamical matrix.

Bertoni et al. [2] have reported first-principle calculations of the lattice dynamics of the *hcp* metals where the total electron energy up to third-order in the electron ion pseudopotential was considered. The authors find that this scheme, which in effect gives rise to three-body forces, leads to the correct ordering of braches at the point K apart form improving the overall agreement with the experiment. The analysis of Bertoni et al. [2] has led to the conclusion that three-body forces are of short-range type, being due to interactions with atoms with few shells. The observations of Brovman et al. [4] and Bertoni et al. [2] have been confirmed by Dasannacharya [1], who has shown that the experimental data on phonon dispersion relations on *hcp* metal along Γ *KM* directions can be explained only when three-body nonpair interactions are incorporated in the theory of lattice dynamics.

It is possible to simulate additional short-range nonpair forces by adding to the long-range matrix an empirical matrix involving general tensor forces extending up to the nearest co-ordination spheres only. Singh and Kushwaha [6] have recently discussed a suitable theoretical model in conjunction with general tensor forces for taking into account the nonpair interaction. This model has given a successful description of the existing experimental data on phonon dispersion curves of *Mg* and *Zn*. The present authors have used the same theoretical model for studying the vibrational properties of *Cd* and *Tl.*

II THEORY

The secular determinant for calculating phonon frequencies ω is written usually as

$$\left| D(\bar{q}) - m\omega^2 I \right| = 0 \qquad \text{... ( )}$$

where $\bar{q}$ is the phonon wave vector, I the unit matrix, m the ionic mass of the atom aod $D(\bar{q})$ is a 6×6 dynamical matrix usually written as

$$D(\bar{q}) = \begin{pmatrix} D(\bar{q}, jj) & D(\bar{q}, jj') \\ D^*(\bar{q}, jj') & D(\bar{q}, jj) \end{pmatrix}$$

where $D(\bar{q}, jj)$ and $D(\bar{q}, jj')$ are 3×3 sub-matrices, $D^*(\bar{q}, jj')$ is the complex conjugate of $D(\bar{q}, jj')$.

In the present work the matrix elements have been regarded as consisting of three terms, first due to electrostatic energy E^c, second due to band structure energy E^{bs} and the third due to short rangc force energy E^{sr}. Therefore one may express

$$D_{\alpha\beta}(\bar{q}) = D^c{}_{\alpha\beta}(\bar{q}) + D^{bs}{}_{\alpha\beta}(\bar{q}) + D^{sr}{}_{\alpha\beta}(\bar{q}) \qquad \text{...(2)}$$

The electrostatc contribution to the elements of the dynamical matrix can be evaluated by standard method [5]. For the evaluation of the numerical values of band structure term we have used the model pseudopotential technique [6]. The nonpair interaction, which is essentially short-range, has been taken into account by considering noncentral forces having tensor character extending upto first and second coordination spheres. The expressions for the matrix elements have been taken from Singh and Kushwaha [6].

III RESULTS AND DISCUSSION

We adpot the formulations of Singh and Kushwaha [6] to calculate phonon dispersion curves. In case of *Cd* and *Tl* the following data corresponding to short-range interaction have been used.

TABLE

Metal	Force constants (10^3 dyn/cm.					
	α_1	α_2	α_3	α_4	β'_1	β_1
Cd	29.32	−9.28	−0.446	13.30	0.521	0.0139
Tl	−3.80	0.476	0.324	−4.916	0.225	−2.565

Using the experessions for the dynamical matrix elements, a Fortran Programme was prepared to compute phonon frequencies for *Cd* and *Tl* along symmetry directions (shown in Figures).

The calculated phonon dispersion curves of *Cd* and *Tl* have been shown in Fig. 1 and Fig. 2 respectively alongwith the experimental points. The calculated frequencies are in good agreement with the experimental results. The present results are better than those obtained earlier by using pairwise forces [7,8]. It is, therefore, evident that unpaired short-range forces must play an effective role in the lattice dynamics of *hcp* metals. As shown by Warren, in the case of *hcp* metals, if only pairwise interaction is present, then

$$2\,{}^{\omega^2}K_5 = {}^{\omega^2}K_1 + {}^{\omega^2}K_3.$$

This is the sum rule of Warren and must be satisfied provided that the interionic forces are in the paired from. Among *hcp* metals, *Mg* is the only example whose phonon frequencies at *K* point satisfy the sum rule of Warren and it does not hold good in the case of other *hcp* metals. The deviation from Warren's sum rule is the direct evidence of the presence of unpaired forces in the crystal. Since the phonon

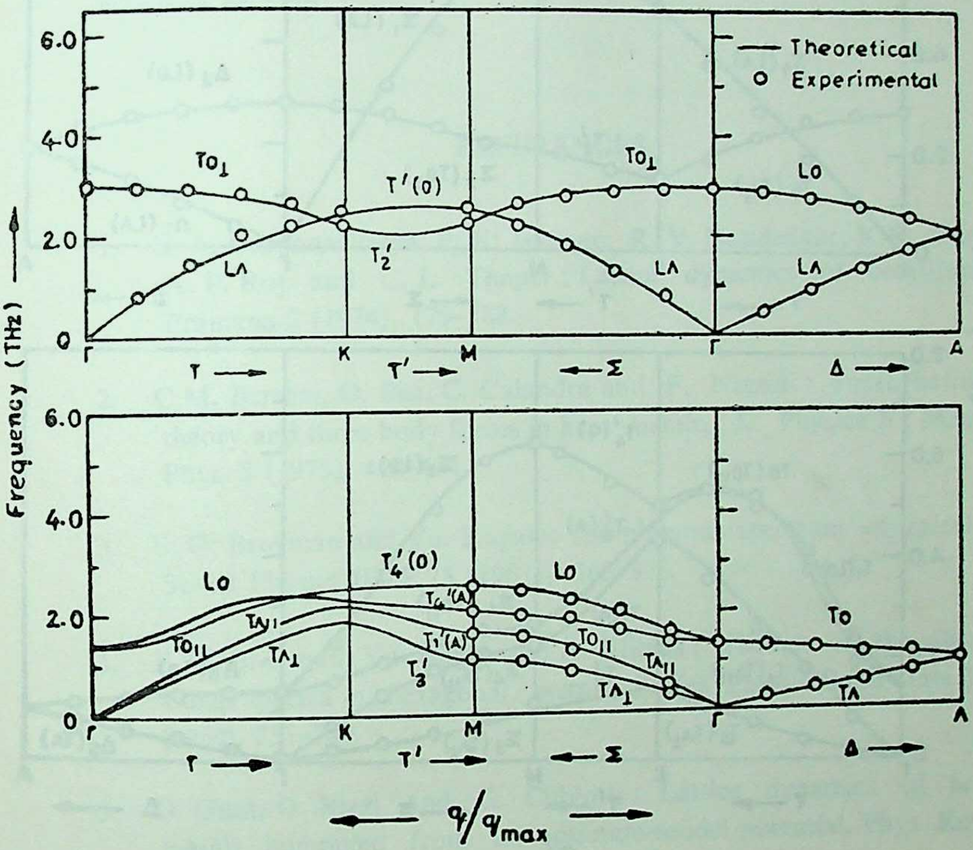

Fig. 1. Comparison of the theoretically computed phonon dispersion relations of *Cd* based on short--range interaction with the experimental results.

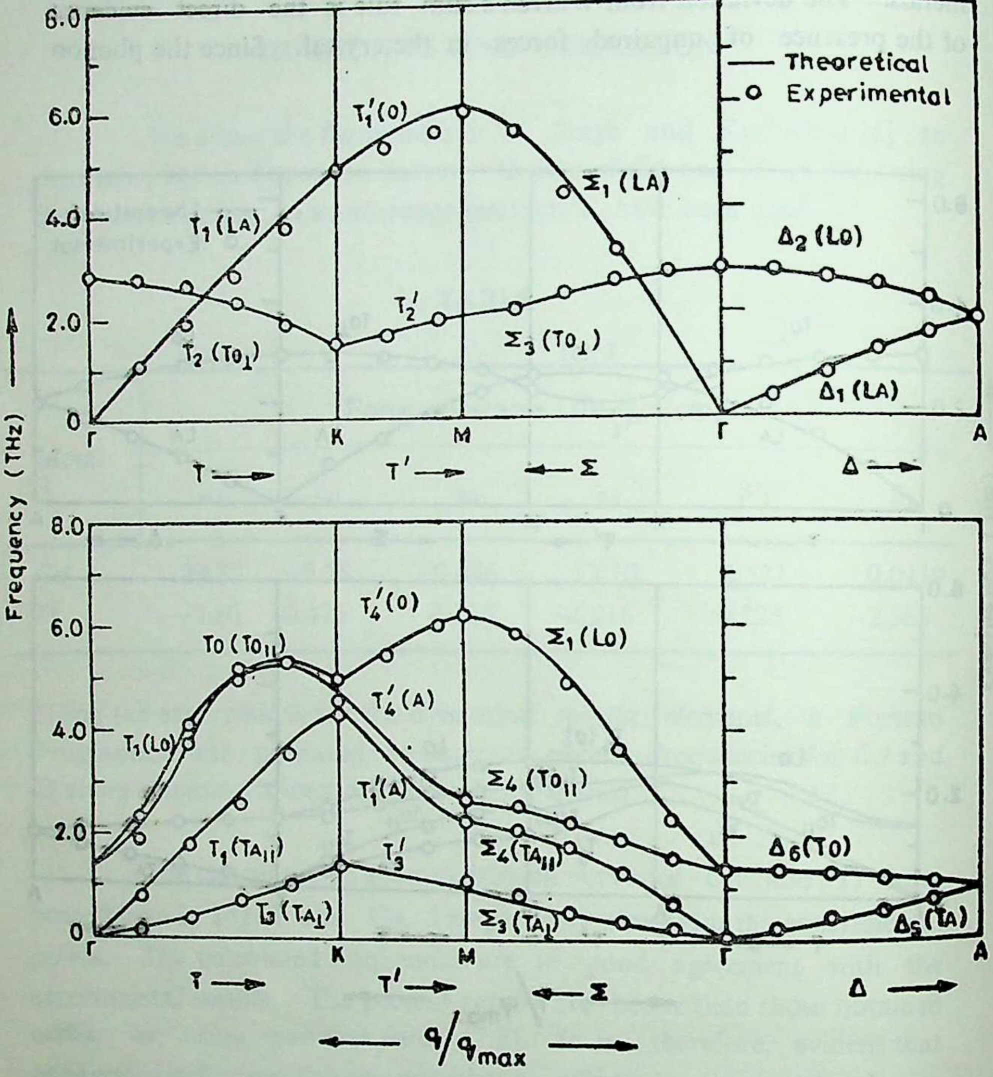

Fig. 2. Comparison of the theoretically computed phonon dispersion relations of *Tl* based on short--range interaction with the experimental results.

frequencies of *Cd* and *Tl* do not satify the Warren's sum rule at *K* point, then it is evident that unpaired short-range forces must play an effective role in the lattice dynamics of *Cd* and *Tl*. It may also be concluded that any calculation which involves pairwise interaction between the ions, whether it is based on the force constant model, or pseudopotential model, is not adequate to explain the phonon frequencies oi *hcp* metals in general.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 1 (1987), 39-52

ION IMPLANTATION AND DEFECT STUDIES IN SEMICONDUCTORS

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ABSTRACT

The ion implantation technique is now well established and widely used to change the electrical, mechanical, thermal or even superconducting properties of any material. The usual restrictions (due to temperature being the only control) of diffusion technique for doping any of the desired elements in semiconductors are nearly nonexistent. One can implant almost any element in any substrate to any desired depth by suitably controlling the energy of the beam. Recent devlopments have made it possible even to form desired thickness of dopants, deep into the bulk the substrate The amount and nature of the damage existing even after the annealing process are easily studied and controlled by the Ruthhrford backscattering/channeling as well as dechanneling experiment. The recent observation of channeling radiation has provided with yet another method to study the defects. A brief review of this work is presented in this article.

Subject Classification : 61. 80Mk

Keywords : Ion implantation, Semiconductors Defects, Channeling. Energy loss.

INTRODUCTION

A suitable amount of foreign atoms, when introduced into an elemental material, tend to change its electrical, mechnaical, chemical, structural and even superconducting properties. The usefulness of diffusion technique for doping is severely restricted by the constraints like solid solubility and limitations of thermal control on the dopant profiles obtained by this method. Ion implantation, on the other hand provides a technique which does not suffer from these above mentioned restrictions. One can, in principle, dope any element into any element. The depth profiles can be suitably controlled by the initial energy given to implant species by the accelerator. In fact it is possible to make alloys at the surface or deep inside the bulk of material. Using high energy implantation (with MeV ions) it is possible to introduce a layer of desired dopant deep into a metal or semiconductor. The semiconductors were the first to be implanted for device fabriccation [8,17] althoguh metals (and even insulators) are now being used for ion implantations [30]. The radiation damage produced during the ion implantation is generally annealed out by thermal or laser annealing. Some times a desired pattern of radiation damange produced with one type of ion can be helpful in radiation enhanced diffusion of second type of ion. The damage produced on soild surface can be very useful in corosion resistance properties and infact in recent years, a suitable chemical impurities doped on the surface combined with controlled amount of radiation damage are being utilized as corosion resistance surface which acts to protect the rest of the specimen.

RANGE THEORY AND ENERGY LOSS

In the theory of ion implantation the main quantity of interest is the range of the ion which is to be calculated from the knowledge of rate of energy loss. Range is defined as the distance travelled by the ion before its energy has fallen below about 20 eV or so, because below those energies, it ceases to move through the solid, becoming trapped by the cohesive forces among the target atoms. Any calculation of range, therefore, requires a knowledge of rate of energy loss in different enesgy regions, so that one can use the formula

$$R_{tot} = \int_{O}^{E} \frac{dE}{(-dE/dx)} \qquad (1)$$

to calculate the total range. The concepts of a projected range R_p (measured parallel to the incident ion direction) and a pependicular range R (measured perpendicular to the incident ion direction) are also of great importance in ion–implantation applications to device fabrication. Thus a comprehensive description of energy loss processes and resulting energy loss rates is at the heart of ion implantation theory.

When an energetic ion moves in a solid and if the velccity is high compared to that of any orbital electron of the ion, then most of the electrons are stripped off and essentially a bare nucleus moves in the solid. As the energetic ion loses energy to the solid through various processess such as excitation and ionization of the target eleetrons, its velocity decreases. As the velocity approaches a velocity equal to that of the electrons in the innermost orbit, the probability of capturing an electrons in that particular orbit increases, whereas the probability of losing an electron from that shell decreases with the result that the electrons are captured in that particular orbit. Similarly, as the ion velocity decreases further and passes through the value of the velocity equal to the orbital velocities of successive outer shells, the electrons are captured successively in those shells, and ultimately at low enough velocities (lower than even the outer most shell electron velocity) essentially a neutral atom moves in the solid. This happens at velocities of the order of $v_0 Z_1^{2/3}$, where Z_1 is atomic number of the incident ion and v_0 is the Bohr velocity. The region of velocities below this velocity is the so called low velocity region where the rate of electronic energy loss is nearly proportional to the projectile velocity. For very low velocities, the so called nuclear stopping starts beeoming important and competes with the electronic stopping. It is these low velocity regions (below $v_0 Z_1^{2/3}$) which are relevant for most of the ion implantation work. Figure 1, shows the behaviour of nuclear and electronic stopping power in different velocity regions. At very low energies, where nuclear stopping become significant, the main effect is dislodgement of atoms from their bonding states and creation of radiation damage of different types in the target material. When the energy of the primary displaced atom is large enough to dislodge more atoms, a damage cascade is induced giving rise to secondary and other higher order drmage cascades. For still lower energies of incident ions (say below 20 eV or so), the chemical binding among the atoms of the target material is more important and the energy is given to the target crystal as a whole: in this case the ion is said to be stopped in the target.

The electronic energy loss in the low velocity region (region III in fig. 1), applicable to most of the ion implantation work is generally calculated either from LSS theory or Firsov theory [12]. The numerical results obtained from these theories have been tabulated for various projectile–target combinations, and are excessively used in range calculations. However, these results do not take into account the effects of single crystalline nature of solids and hence need to be modified ion implantation profiles in single crystal silicon and germanium are required to be estimated. Of specially importance are Z_1 and Z_2 variations in electronic stopping power [13,19] which are very pronounced whenever the projectile is steered along one of the major crystallographic directions [3,9,11]. These channeling situations occur whenever an ion's velocity direction makes an angle smaller than the Lindhard critical angle [15] ψ_c with respect to a major crystallographic dircetion. This critical angle is given by

$$\psi_c = \sqrt{\frac{2 Z_1 Z_2 e^2}{dE}} \tag{2}$$

where Z_1 and Z_2 are the atomic numbers of projectile and target respectively, E is the projectile energy and d is interatomic distance along the relevant axial direction. The corresponding critical angle for planar channeling is

$$\psi_p = \sqrt{2\pi N d_p Z_1 Z_2 e^2 a/E} \quad (3)$$

where d_p is the interplanar density. N is the bulk density of aotms in the target and a is the Thomas—Fermi screening radius given by

$$a = 0.8853 \quad a_0/(Z_1^{2/3} + Z_2^{2/3})^{1/2} \quad (4)$$

where a_0 is the Bohr radius. The earlier explanations of Z_1 — oscillations [2,6,10,34] where based on inclusion of electron shell structure effects in the overlap density of projectile and target atoms. This was done by replacing the Thomas–Fermi electron density by squares of Hartree–Fock wave functions of individual shells. The above modifications did yield oscillatory behaviour in electronic stopping but not as much as experimentally observed. Moreover, the calculated oscillations show severe damping effect for higher Z_1 values, whereas experimentally there is very little damping. An alternative model [4,5] was proposed in which the stopping power oscillations are dirctly related to similar oscillations in the momentum transfer (or diffusion or transport) cross section for scattering of target electrons (assumed to be free) in the potential field of projectil atom.

Thus one gets

$$\frac{-dE}{dE} = nm\ v^2\ Q_d \quad (5)$$

where

$$Q_d = \frac{4\pi}{k^2} \sum_l (l+1)\ Sin^2 (n_l - n_{l+1}) \quad (6)$$

is the momentum transfer cross section, k being the wave number of electron being scattered in the centre of mass frame, n_l the phase shift induced by the field of the atom in the l^{th} partial wave of the expansion of the electron wave function, and n is the effective electron density sampled by the moving projectile atom. The phase shifts n_l are obtained by comparing the asymptotic solutions of the radial part of the Schrodinger eq

$$\frac{d^2G_l}{dr^2} + [k^2 + U(r) - \frac{l(l+1)}{r^2}]\ G_l = 0 \qquad (7)$$

with and without the interaction potential field $U(r)$. It is found that the momentum transfer cross-section, even for a statistical potential of Moliere-type shows oscillatory dependence [4,5] in Q_d. The Thomas-Fermi numbers were fitted to a form $U(r) = \sum_{i=1}^{3} a_l \frac{e^{-b_1 r}}{r}$ where a_l and b_l were obtained by a least square fit. With this form of potential we got oscillatory dependence with maxima and minima at the same Z values as in stopping power curves. A detailed calculation for the effective electron density (with detailes of atomic shell structure included) yields reasonably good agreement of silicon stopping power [4,5]. Furthermore, for higher values of Z_l where data on metallic targets (gold and tungsten) is available, the theoretical predictions based on above model yield equally pronounced [21˙ oscillations as for lower Z_l. The variation of stopping power with the atomic number of solid target (Z_2) comes from the corresponding variation of effective electron density [22,23], although in gaseous targets, both Z_2 and Z_1 oscillations come from the same physical basis namely the variations in the momentum transfer cross section.

It has been found [31,32], that for solid targets, the Z_1 oscillations start diminshing with increasing projectile velocity and are completely washed out beyond 3 a.u. velocity approximately. On the other hand Z_2 variations persist upto about 8 a. u. projectile velocity. These experimental findings are consistent [23] with the above theoretical model [4,5]. The stopping powers for still higher velocities fall essentially in the region I of figure 1 and usual Bethe-Bloch theories with suitable modifications for crystallographic effects are to be used [24]. However most of the ion implantation work falls in the lower velocity region discussed above.

3. STUDY OF DEFECTS

The energetic charged particle probe is most powerful technique for studying the defects is any solid material, specially in crystalline substance. As illustrated in figure 3, it is possible to pin point

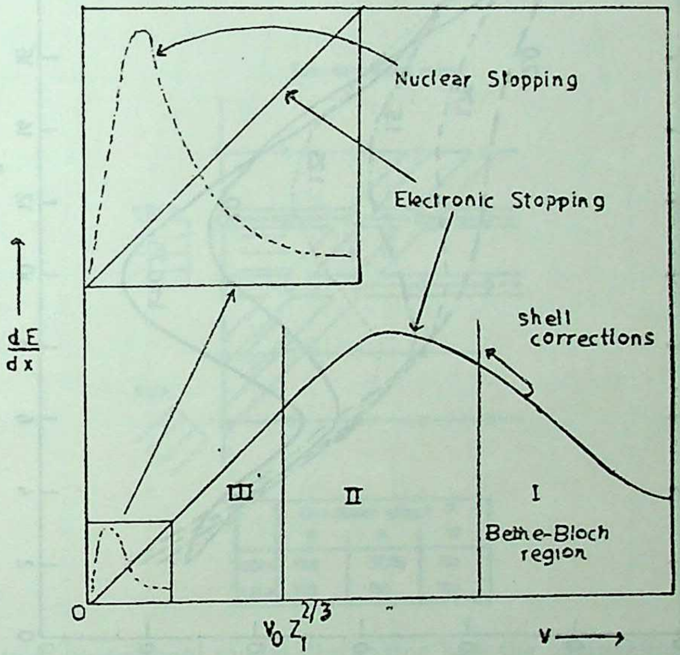

Nuclear Stopping
Electronic Stopping
shell corrections
dE/dx
III
II
I
Bethe-Bloch region
O
$V_0 Z_1^{2/3}$
V

Fig. 1. Schematics of different enrrgy loss regions. These curves are qualitative only and coordinates are not to the scale. The solid curve represents electronic energy loss and the dotted curve represents nuclear energy loss.

the exact position of point defects by studying their yields (either Rutherford backscattering or the transmission) In this case, as also for stacking faults [14, 28], the main effect is *obstruction* of the propagating projectile by the defects. This is to be constrated with the

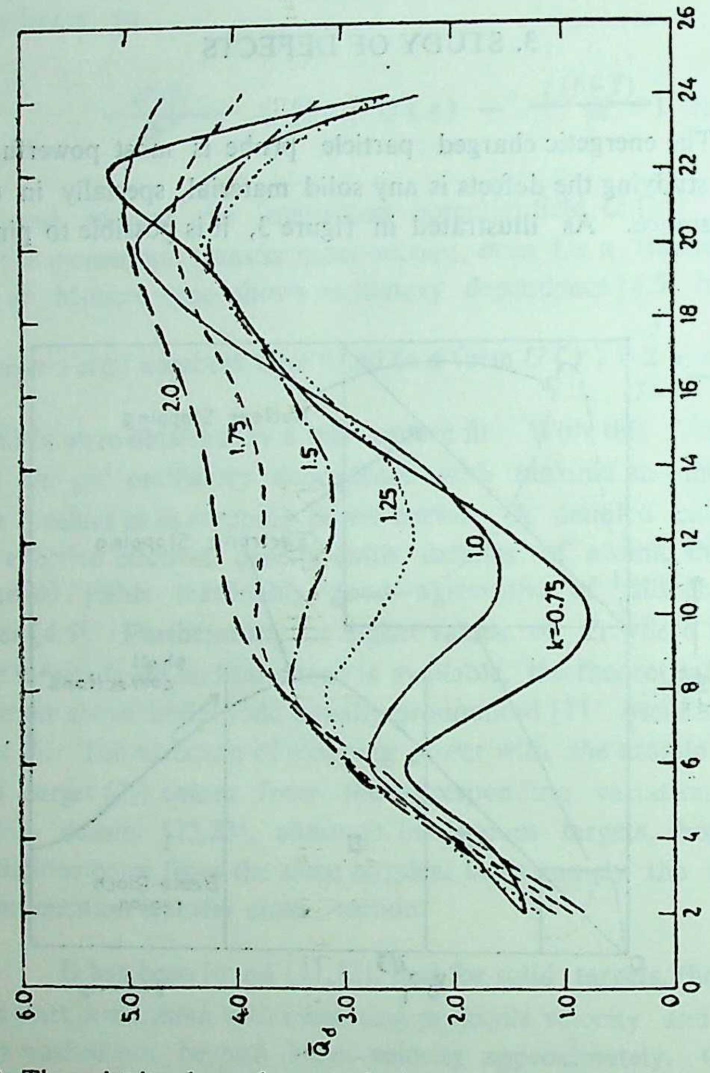
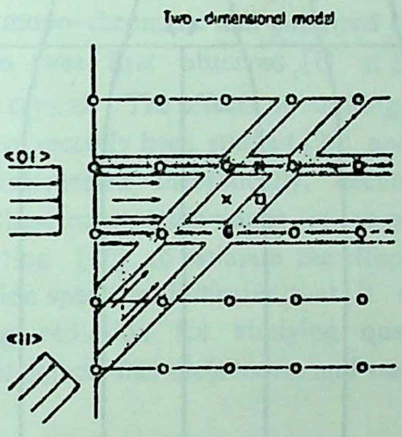


Fig 2. The velocity dependence of Z_1 oscillations in the momentum transfer cross section (and hence in stopping power) for $k=0.75$ to 2 au. (The higher Z_1 oscillations are also washed out by about 3 au).

distortion effects introduced in the channels by the deffects like dislocations, as illustrated in figure 4. The dechanneling cross section due to these various kinds of defects gives direct information about the nature and concentration of defects. For instance, the interstitials give rise to potential in which the channeled particle is scattered so that the transverse energy $E_\perp$ changes. If the new $E_\perp$ is greater than the critical transeverse energy $E\psi$ [8], the particle is dechanneled. A detailed calculation [14,25,28] shows that the dechanneling cross section due to interstitials varies as $E^{-1/2}$ with particle energy. For stacking faults, the corresponding dechanneling cross section is independent of energy. For stacking faults, the corresponding dechanneling cross section is independent of energy. The detailed calculations have been summarized in a recent article [25].



	Directional effect ? ●	×	□
<01>	Yes	50%	No
<11>	Yes	No	Yes

Fig. 3. A two dimensional model illustrating how the channeling effect may be used to locate foreign atoms in a crystal As shown by the table three possible sites for a foreign atom (O, II and X) can readily be distinguished by comparing the channeling behaviour along the <01> and along <11> directions.

The situation is entirely different for dislocations. The channels affected by the dislocations are distorted and the channeled parti-

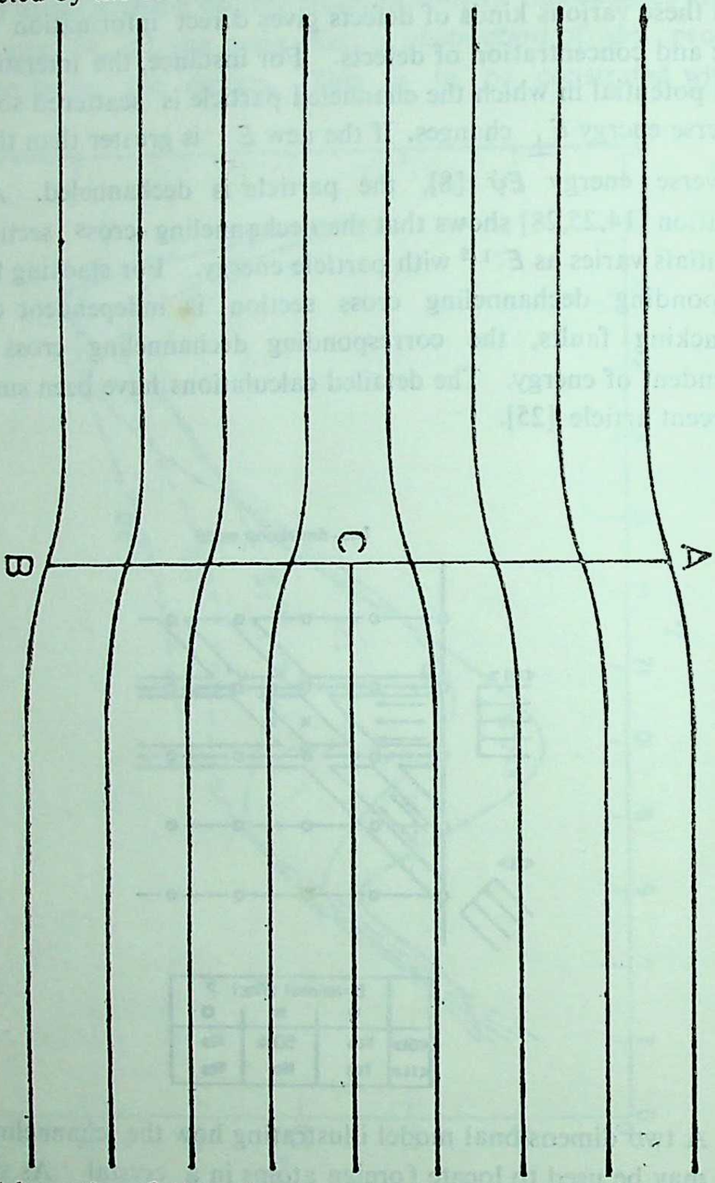


Fig. 4. Schematics of distortion introduced in the channels in the vicinity of a dislocation core C. The distortions in opposite sides of AB are in opposite directions. The distortions above and below the core C are also in opposite directions.

lce is influenced by a transverse centrifugal force whose magnitude($2E/R$) depends on the curvature of the channel which in turn varies with distance from the dislocation core. Whether or not the particle is dechanneled, depends upon the relative magnitudes of the centrifugal force and the restoring force duc to the planar or axial potential. A detailed analysis [14,25,28] shows that the dechanneling cross section here varies as $E^{1/2}$ for straight dislocations. All this work including the effects on stopping power due to these various kinds of defects has been discussed in a review article by the author [25].

Recently another intersting technique has shown great promisc in defects studies. This technique however depends upon availability of relativistically fast electrons or positrons. Under channeling conditions, the relativistic electrons and positrons are found to emit what is called channeling radiation. This radiation, also known as Kumakhov radiation, is intense, mono-chromatic and polarized (for planar channeling), This radiation was first observed [1] at Stanford in 1979 in perfect silicon single crystal. The effects of stacking faults and dislocation in diamond have recently been studied [20] and the effects observed are still being interpreted theoretically, because the amount and nature of deffects in these natural diamonds is very complicated. A simple model calcuation [26] to estimate the effects of dislocation on the channeling radiation spectrum indicated that it should be possible to use the channeling radiation for studying quantitatively, the low concentration of these defects like dislocation and stacking faults.

4. CONCLUSIONS

The ion implantation technique provides with much more control on doping profiles than is possible with other conventional techniques. The limitations in other techniques due to solid solubilities are overcome in a natural way in ion implantation. The major problem related with this useful technology is annealing of defects and damage created during the ion implantation. This is overcome partly by using thermal or laser annealing, The study and identification of defects in doped samples is easily carried out by using the enrgetic ion beams and performing dechanneling experiments and/or Rutherford backscattering/channeling experiments.

The recent discovery of channeling radiation holds great promise in defect studies and current theoretical and experimental work at various laboratories needs further impetus.

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COINCIDENCE POINT THEOREMS FOR MULTIVALUED MAPPINGS IN MENGER SPACES

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1. ABSTRACT

The problem of the existence of a coincidence point for two multivalued mappings of a Menger space is considered in this paper. These mappings are assumed to satisfy some inequalities or implications which are generalisation of the conditions given in [2], [3], [4].

Key words and phrases : Coincidence point, Menger space.

AMS (MOS) Subject Classifications (1980) : 54H25, 47H10.

2. INTRODUCTION

K. Menger initiated the study of probabilistic metric spaces in 1942 [6]. The subject began to grow rapidly due to the work of Schweizer, Sklar, Throp and the others after 1962.

The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. Some fixed point theorems for mappings in a probabilistic space were proved by several authors : G. Bocsan, G. Constantin, A. T. Bharucha-Reid, V. M. Sehgal, S. L. Singh, O. Hadzic, S. Chang, M. Stojaković (cf. [9]-[12]). The problem of the fixed point for multivalued mappings was investigated by O. Hadzić [2]-[3].

3. PRELIMINARIES

Let R denote the reals and $R^+ = \{x \in R : x \geq 0\}$. A mapping $F : R \to R^+$ is called a *distribution function* if it is nondecreasing left continuous with $\inf F = 0$ and $\sup F = 1$. We shall denote by $\triangle$ the set of all distribution functions.

A commutative, associative and nondecreasing mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is a *T–norm* iff $t(a, 1) = a$ for all $a \in [0, 1]$ and $t(0, 0) = 0$.

DEFINITION 1. A *Menger space* is a triplet $(X, \mathfrak{F}, t)$, where X is an abstract set of elements, $\mathfrak{F}$ is a mapping of $X \times X$ into $\triangle$ and t is a T-norm. We shall denote the distribution function $\mathfrak{F}(x, y)$ by $F(x, y)$ and $F(x, y; \varepsilon)$ will represent the value of $F(x, y)$ at $\varepsilon \in R$. The functions $F(x, y)$ $x, y \in R$, are assumed to satisfy the following conditions :

1. $F(x, y; \varepsilon) = 1$ for all $\varepsilon > 0$ iff $x = y$.
2. $F(x, y; 0) = 0$, for all $x, y \in X$.
3. $F(x, y) = F(y, x)$ for all $x, y \in X$.
4. $F(x, y; \varepsilon + \delta) \geqslant t(F(x, z; \varepsilon), F(z, y; \delta))$ for all $x, y, z \in X$ and all $\varepsilon, \delta \in R^+$.

Throughout this paper H will always denote the specific distribution function defined by

$$H(\varepsilon) = \begin{cases} 0 & \varepsilon \leqslant 0 \\ 1 & \varepsilon > 0. \end{cases}$$

The concept of neighbourhoods in a Menger space was introduced by Schweizer and Sklar [8]. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0, 1)$, then (ε, λ) – neighbourhood of x, denoted by $u_x(\varepsilon, \lambda)$, is defined by

$$u_x(\varepsilon, \lambda) = \{ y \in X : F(x, y; \varepsilon) > 1-\lambda \}.$$

If t is continuous, then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family $\{u_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda \in (0, 1)\}$ of neighbourhoods.

A random normed space is a Menger space, where X is a real or complex linear space and the mapping $\mathcal{F}(x, y) = F(x-y)$ satisfies the next conditions :

1. if s is a non-zero scalar, then

$$F(sx; \varepsilon) = F(x; \varepsilon/|s|) \text{ for all } x \in X \text{ and all } \varepsilon > 0.$$

2. $t(a, b) \geqslant \max\{a + b - 1, 0\}$ for all $a, b \in R^+$.

Let X be a topological vector space, $M \subset X$. The mapping $A : M \to 2^M$ is *f-strongly demicompact*, where $f : M \to M$, if for every sequence $\{x_n\}_{n \in N} \subset M$, for which the sequence $\{fx_n - y_n\}_{n \in N}$, $y_n \in Ax_n$ is convergent, there exists a subsequence $\{fx_{n_k}\}_{k \in N}$ which is convergent.

If t is a T-norm, then we can define the mapping

$$t_n(a) = \underbrace{t(\ldots t(t(t(a, a), a), \ldots\ldots, a)}_{\text{n-times}}.$$

0. Hadzic introduced a class of T-norms which have the property that the family $\{t_n(a)\}_{n \in N}$ is equicontinuous at the point $a = 1$.

Now we can introduce the notion of a weakly comuting mapping for a multivalued mapping $A : X \to 2X$ and a single-valued mapping $f : X \to X$. Two mappings A and f are *weakly commuting* if $f(Ax) \subset A(fx)$ for all $x \in X$.

It is obvious that two commuting mappings also commute weakly, but two weakly commuting mappings are not necessarily commuting, as it is shown in the example given below.

Example. Let $X = R^+$ with usual metric and let $f : R^+ \to R^+$, $f(x) = \ln(x+1)$ and $A : R^+ \to 2^{R^+}$, $A(x) = (x, x+1)$. Then

$$f(Ax) = f(x, x+1) = (\ln(x+1), \ln(x+2))$$

$$A(fx) = A(\ln(x+1)) = (\ln(x+1), \ln(x+1)+1).$$

Since $\ln(x+2) < \ln(x+1)+1$ for all $x \geqslant 0$, it implies that

$$f(Ax) \subset A(fx) \qquad \text{and} \qquad f(Ax) \neq A(fx).$$

4. FIXED POINT THEOREMS

THEOREM 1. Let $(X, \mathfrak{F}, t)$ be a complete Menger space with T-norm t such that $\sup_{a<1} t(a, a) = 1$, S a closed subset of X, f a continuous mappnig from S to S, $A : S \to 2^{f(S)}$ and $B : S \to 2^{f(S)}$ closed mappings such that for every $u, v \in S$ every $x \in Au$ and every $\delta > 0$ there exists $y \in Bv$ such that

$$\text{(1)} \qquad F(x,y\,; \varepsilon) \leqslant F(fu, fv;\ (\sigma-s)/q)$$

for every $\varepsilon > 0$, where $q \in (0, 1)$. If A and B are weakly commuting with f and if the family $\{t_n(a)\}_{n \in N}$ is equicontinuous at the point $a = 1$, then there exists $z \in S$ such that $fz \in Az \cap Bz$.

PROOF. First, we shall show that we can form the sequence $\{fx_i\}_{i \in N} \subset S$ with the property that

$$F(fx_i,\ fx_{i+1}\,; \varepsilon) \geqslant F(fx_1, fx_2\,; (\varepsilon - iqi)/qi) \text{ for every } i \in N.$$

For every $x_1 \in S$ there exists $x_2 \in S$ such that $fx_2 \in Ax_1$ and there exists $y_2 \in Bx_2$ such that for $\delta = q$ (from (1))

$$F(fx_2, y_2;\ \varepsilon) \geqslant F(fx_1,\ fx_2;\ (\varepsilon - q)/q)$$

Further, let $x_3 \in S$ be such that $fx_3 = y_2$ and let $y_3 \in Ax_3$ be such that for $\delta = q^2$ (from (1))

$$F(fx_3, y_3; \varepsilon) \geqslant F(fx_2, fx_3; (\varepsilon - q^2)/q) \geqslant F(fx_1, fx_2; ((\varepsilon - q^2)/q - q)/q)$$
$$= F(fx_1, fx_2; (\varepsilon - 2q^2)/q^2).$$

Continuing this procedure we get that

(2) $$F(fx_j, fx_{j+1}; \varepsilon) \geqslant F(fx_1, fx_2; (\varepsilon - i\, q^i)/q^i).$$

In order to prove that the sequence $\{fx_i\}_{i \in N} = \{y_i\}_{i \in N}$ is convergent we proceed as follows :

From $\varepsilon(1 - \frac{1}{j+1}) = \varepsilon(\frac{1}{1.2} + \cdots + \frac{1}{j(j+1)})$ for all $j \in N$, we get

$F(y_i, y_{i+j}; \varepsilon) \geqslant F(y_i, y_{i+j};\ \varepsilon(1 - 1/j+1)) \geqslant$

$\underbrace{t(t \ldots t}_{j\text{-times}} (Fy_{i+j}, y_{i+j-1};\ \varepsilon/j(j+1)),\ F(y_{i+j-1}, y_{i+j-2};\ \varepsilon/(j-1)j), \ldots\ldots,$
$F(y_{i+1}, y_i; \varepsilon/1.2))$

$\geqslant t(t \ldots t)\ F(y_1, y_2;\ (\varepsilon/j(j+1) - (i+j-1)q^{i+j-1}) / q^{i+j-1}),\ F(y_1, y_2;$

$(\varepsilon/(j-1)j - (i+j-2)q^{i+j-2}) / q^{i+j-2}), \ldots\ldots,$
$F(y_1, y_2; (\varepsilon / 1.2 - i\, q^i) / q^i)) \geqslant t_j (F(y_1, y_2; (\varepsilon / j\,(j+1) - iq^i)/q^i)).$
Since $\lim_{i \to \infty} (\varepsilon / j(j+1) - iq^i) / q^i = \infty$ for all $\varepsilon > 0$ and all $j \in N$, from the equicontinuity of $\{t_j(a)\}_{j \in N}$ at the point $a = 1$ we get that the sequence $\{y_n\}$ is a Cauchy sequence, that is, for all $\varepsilon > 0$ and all $\lambda \in (0, 1)$ there exists $i_0 \in N$ such that
$F(y_i, y_{i+j}; \varepsilon) > 1 - \lambda$ for all $i > i_0$ and all $j \in N$.

The space $(X, \mathfrak{F}, t)$ is complete, which means that $\lim_{n \to \infty} y_n = \lim_{n \to \infty} fx_n = z \in S$. Further, from the next two implications

$$fx_{2n} \in Ax_{2n-1} \Rightarrow ffx_{2n} \in f\,Ax_{2n-1} \subset Afx_{2n-1},$$

$$fx_{2n+1} \in Bx_{2n} \Rightarrow ffx_{2n+1} \in f\,Bx_{2n} \subset Bfx_{2n},$$

when $n \to \infty$ we get $fz \in Az \cap Bz$, what we had to prove.

THEOREM 2. Let $(X, \mathfrak{F}, t)$ be a complete random normed space with T-norm t such that $\sup_{a<1} t(a, a) = 1$, S be a closed subset of X, f a continuous self-mapping of S, $A : S \to 2^{f(S)}$ and $B : S \to 2^{f(S)}$ be closed mappings such that for every $u, v \in S$, every $x \in Au$ and every $\delta > 0$ there exists $y \in Bv$ such that

$$F(x - y; \varepsilon) \geq F(fu - fv; (\varepsilon - \delta)/q)$$

for every $\varepsilon > 0$, where $q \in (0, 1)$. If f is weakly commuting with A and B and if A or B is f-strongly demicompact, then there exists $z \in X$ such that $fz \in Az \cap Bz$.

PROOF Since every random normed space is a Menger space, from (2) we get

$$\lim_{i\to\infty} F(fx_i - fx_{i+1}; \varepsilon) \geqslant \lim_{i\to\infty} (fx_1 - fx_2; (\varepsilon - iq^i) / q^i) = 1$$

for all $\varepsilon > 0$, which means that $\lim_{i\to\infty}(fx_i - fx_{i+1}) = 0$. If the mapping A is f-strongly demicompact then, from the last equality and from the fact that $fx_{2n} \in Ax_{2n-1}$, we get that there exists a convergent subsequence $\{fx_{n_k}\}_{k\in N} \subset \{fx_n\}_{n\in N}$. Let $\lim_{k\to\infty} fx_{n_k} = z$ Since $\lim_{k\to\infty} (fx_{n_k} - fx_{n_k} + 1) = 0$, it implies that $\lim_{k\to\infty} fx_{n_k - 1} = z$ and $\lim_{k\to\infty} fx_{n_k + 1} = z$. Continuing this process, we get that $\lim\limits_{k\to\infty} fx_{n_k+i}=z$ for all $i\in N$. Let $\{fx_{m_i}\}_{i\in N}$ be the subsequence of $\{fx_n\}_{n\in N}$ with odd index, such that $\lim\limits_{i\to\infty} fx_{m_i}=z$. From the implication

$$fx_{m_i}\in Ax_{m_{i-1}} \Rightarrow ffx_{m_i} \in fAx_{m_{i-1}} \subset Afx_{m_{i-1}},$$

letting $i\to\infty$, we get $fz \in Az$.

The proof that $fz \in Bz$ is analogous.

0. Hadzic, following the results of T. Hicks [5] introduced the notion of generalized h-contraction. In the next theorem we shall give a generalization of this result for the case of two multivalued mappings.

Let $X, F, t)$ be be a Menger space and $S,\ T \in CB\ (X)$, where $CB(X)$ is the set of all closed and bounded subsets of X. Then for every $\varepsilon>0$ one can define $F(S, T; \varepsilon)$ by :

$$F(S, t; \varepsilon)=\min\{\inf_{x\in S}\ \sup_{y\in T} F(x, y; \varepsilon)\ \ \inf_{y\in T}\ \sup_{x\in S} F(x, y; \varepsilon).$$

Let h: $R^+\to R^+$, $S \subset X$, $f{:}S\to S$, $A : S \to CB(fS)$, $B : S \to CB(fS)$, and A and B are continuous mappings and let for every $\varepsilon>0$ and every $x, y\in S$ the next implication holds

$$F(fx, fy; \varepsilon) > 1 - \varepsilon \to F(Ax,\ By;\ \ h(\varepsilon)) > 1 - h(\varepsilon). \tag{3}$$

V. Radu in [7] proved, for the T-norm $t(a, b) \geqslant t_m\ (a, b) = \max\ \{ a + b - 1,\ 0\}$, that the (ε, λ)-topology in a Menger space $(X, \mathfrak{F}, t)$ is equivalent to the topology induced by the metric defined by: $d(x, y)\ = \sup\{\varepsilon;\ \ (x, y; \varepsilon) < 1\text{-}\varepsilon\}$. Using all previous results and notions we we can prove the next theorem.

THEOREM 3. Let $(X, \mathfrak{F}, t)$ be a compact Menger space with T-norm $t \geqslant t_{u}$ and S, A and B be as defined above. If implication (3) is satisfied for the mapping $h : R^{+} \to R^{+}$ which is right continuous and $h(t) < t$ for all $t \in R^{+}$ then there exists $x \in S$ such that $x \in Ax \cap Bx$.

Proof. First, we shall prove that the mappings A, B and f satisfy the inequality

$$(4) \qquad D(Ax, By) \leqslant h\,(d(x, y))$$

for all x, $y \in S$, where D is the Hausdorff metric:

$$D(A, B) = max\{\sup_{x \in A} \inf_{y \in B} d(x, y),\ \sup_{y \in B} \inf_{x \in A} d(x, y)).$$

In order to prove that (4) holds, we shall prove that the inequality $d(x. y) < a$ implies that $D(Ax, By) < h(a)$. If $d(x, y) < a$, then $F(x, y; a) > 1 - a$ and from (3):

$$(Ax, By; h(a)) > 1 - h(a),$$

that is,

$$\inf_{u \in Ax} \sup_{v \in By} F(u, v; h(a)) > 1 - h(a) \quad \text{and}$$

$$\inf_{u \in By} \sup_{v \in Ax} F(u, v;\ h(a)) > 1 - h(a).$$

From the last two inequalities we get that for every $u \in Ax$

$$\sup_{v \in By} F(u, v; h(a)) > 1 - h(a)$$

and for every $u \in By$

$$\sup_{v \in Ax} F(u, v\,;\ (h(a)) > 1 - h(a).$$

Hence for every $u \in Ax$ there exists $v(u) \in By$ such that

$$F(u, v(u); h(a)) > 1 - h(a), \text{ that is } d(u, v(u)) < h(a).$$

Now we have that $\sup_{u \in Ax} \inf_{v \in By} d(u, v) < h(a)$ and, similarly,

$\sup_{u \in By} \inf_{v \in Ax} d(u, v) < h(a)$.

From the definition of the Hausdorff metric and last inequalities, we get $D(Ax, By) < h(a)$ and from right continuity of the mapping h we get that $D(Ax, By) < h(d(x, y))$.

Theorem 3 from [1] completes the proof.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. 1 (1987), 63-66

PROXIMITY EXTENSIONS

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ABSTRACT

It is shown, by means of an example, that Leader's construction for the proximity extension (the Smirnov compactification) of an Efremovič proximity space fails for a Lodato proximity space. However, a recent construction, suggested by Bentley and modified by Gagrat and Thron, which uses nearness and contiguity structures, provides a method of getting a proximity extension of a Lodato space. This agrees with Leader's extension in the case of an Efremovič proximity.

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INTRODUCTION

One of the most elegant constructions of the Smirnov compactification of an Efremovič (or EF-) proximity space was discovered by Leader [5]. This consists of constructing the proximity space X* of all clusters of a given EF-space (X, π) on which the proximity is defined by : two subsets α, β of X^* are proximal or near provided if A, B are subsets of X, A absorbs α (i. e. A is a member of every cluster in α) and B absorbs β, then $A \in \pi(B)$. It was shown in [2] that (a) a family σ of subsets of an EF-space X is a cluster if and only if it is a maximal bunch, and (b) the space of all maximal bunches of a separated Lodato (or Lo-) space together with the absorption topology is a T_1-compactification which coincides with the Leader's construction if the given proximity is EF. Finally, Thron [9] proved that every separated LO-proximity is induced by the elementary proximity on the above T_1-compactification.

The question arises : can we define a LO-proximity directly on the space of maximal bunches? We construct an example to show that Leader's construction does not work in this case. However, we can show that a method proposed by Bentley [1] (to construct a contiguity extension of a contiguity space) and modified by Gagrat and Thron [3], enables us to define the elementary proximity directly by means of a compatible contiguity on the base space.

For information on proximity spaces see [2], [5], [6] and [8], on contiguity spaces [4] and on nearness structures [3], [7] where further references will be found. We will follow the terminology of [3] for the most part.

EXAMPLE

We construct an example of a LO-space S for which some maximal bunches are not clusters, and Leader's construction does not yield a proximity on the family of all maximal bunches.

First we give an example of Ward [10]. Consider the T_1-space which consists of three points a, b, c, and three infinite sequences (x_n), (y_n), and (z_n) such that $\{b, c\}$, $\{c, a\}$, $\{a, b\}$ are the set of limit points of (x_n), (y_n), and (z_n) respectively. Let this space be given the elementary proximity. Then the family of all infinite subsets of the space is a maximal bunch which is not a cluster, since the set $\{a, b, c\}$ is near every member of the family but is not in the family.

Now we construct the T_1-space S by taking the topological sum of countably many copies of the Ward space. Let $A = \{a_i : i \in N\}$, $B = \{b_i : i \in N\}$, $C = \{c_i : i \in N\}$, $X_i = \{f_i(n) : n \in N\}$, $Y_i = \{g_i(n) : n \in N\}$, $Z_i = \{h_i(n) : n \in N\}$, $X = \cup \{X_i : i \in N\}$, $Y = \cup \{Y_i : i \in N\}$, $Z = \cup \{Z_i : i \in N\}$. Let $S = A \cup B \cup C \cup X \cup Y \cup Z$ where A, B, C are mutually disjoint and the set of limit points of X_i, Y_i and Z_i are $\{b_i, c_i\}$, $\{c_i, a_i\}$, $\{a_i, b_i\}$ respectively. Assign the elementary proximity π_0 to S.

It is easily verified that for each $i \in N$, $\sigma_i = \{P \subset S : P \cap (X_i \cup Y_i \cup Z_i)$ is infinite$\}$ is a maximal bunch which is not a cluster. Let $\sigma_x = \{P \subset S : P \in \pi_0(x)\}$ be the point cluster at x. Now consider three families of maximal bunches, $\alpha = \{\sigma_{a_i} : i \in N\}$, $\beta = \{\sigma_{b_i} : i \in N\}$ and $\gamma = \{\sigma_i : i \in N\}$. Clearly X absorbs γ, A absorbs α and X is not in $\pi_0(A)$; so, by Leader's definition, γ is far from α. Similarly, by considering Y and B, it follows that γ is far from β. However, γ is not far from $(\alpha \cup \beta)$, since if P absorbs $\alpha \cup \beta$, then $P \supset A \cup B$ which absorbs γ. Thus the union axiom fails and Leader's definition does not yield a proximity on the family of maximal bunches. (We note that Leader's proof of the union axiom for the space of clusters uses the Efremovič axiom.)

CONTIGUITIES AND EXTENSIONS

Suppose now that (X, π) is a separated LO-space and that X^* is the space of all maximal π-bunches with the absorption topology. Define the contiguity ξ^π on X by $\mathcal{U} \in \xi^\pi$ if and only if $|\mathcal{U}| < \aleph_0$ and $\mathcal{U} \subset \sigma$ for some $\sigma \in X^*$ (see Gagrat and Thron [3]). Since (i) every $\xi\pi$ - bunch is a π-bunch and (ii) every $\sigma \in X^*$ is a

ξ^π - bunch, it follows that X^* is also the set of all maximal ξ^π -bunches.

A LO-proximity π^* can be defined on X^* by : for subsets α, β of X^*, $\beta \in \pi^*(\alpha)$ if and only if $(\cap \alpha) \cup (\cap \beta)$ is ξ^π - compatible, that is, if $\mathcal{U} \subset (\cap \alpha) \cup (\cap \beta)$ and $|\mathcal{U}| < \aleph_0$, then $\mathcal{U} \in \xi^\pi$ (see [3]). π^* is, in fact, the elementary proximity on X^*. To see this let $\beta \in \pi^*(\alpha)$. Then $(\cap \alpha) \cup (\cap \beta)$ is ξ^π- compatible and is therefore contained in a maximal ξ^π- compatible family σ which is a ξ^π - clan (Theorems 2 2, 2.5 in [3]). Obviously $\sigma \in Cl\,\alpha \cap Cl\,\beta$ in the absorption topology; so π^* is elementary.

We note that in the example considered earlier, $\{X, Y, A, B\} \subset (\cap (\alpha \cup \beta)) \cup (\cap \gamma)$ but $\{X, Y, A, B\}$ is not in ξ^π so $\alpha \cup \beta$ is not in $\pi^*(\gamma)$.

If π is EF, then X^* is the space of all π–clusters and π^* is also an EF-proximity on X^*.This shows that as far as EF-proximities are concerned, contiguities are unnecessary for proximity extensions. (see also [7].)

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol.1 (1987), 67-68

DETERMINATION OF SPECIAL PALINDROMIC PRIME NUMBERS

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ABSTRACT. In this paper we give a complete set of palindromic prime numbers of some type. The calculation was done by a small size computer.

There are about 5761455 prime numbers between 1 and 10^8, and 455052512 prime numbers between 1 and 10^{10}. Therefore it is impossible to make the table of prime numbers between 1 and 10^{10}, Consequently we are interested in special forms of prime numbers.

A natural number is called palindromic, if the numeral is identical with its reverse. For examle, 100242001 is palindromic. To denote large numbers, we shall use the Yates–Gernedl notation. If a digit appears s-times, We write subscript s after the digit. For example, 10000200001 is denoted by $10_4 20_4 1$.

We decide all palindromic prime numbers of the form $y0_n \times 0_n y$ ($y = 1, 3, 7, 9$; $x = 0, 1 \ldots\ldots, 9$ and $n = 0, 1, \ldots\ldots\ldots, 10$).

Palindromic prime numbers with 3 digits are
101, 131, 151, 181, 191, 353, 373, 383, 727, 787, 797.

Palindromic prime numbers with 5 digits are
10301, 10501, 10601, 30103, 30203, 30403,
30703, 30803, 70207, 70507, 70607, 70709.

These numbers are also seen from some prime number tables. However. the following results are obtained

(1) The case of $10_n \times 0_n1$.

101, 131, 10301, 1003001, 100030001, $10_{10}30_{10}1$,
151, 10501, 100050001, 10000500001, $10_7 50_7 1$,
10601, 100060001.
181, 1008001, 1000008000001, $10_8 80_8 1$,
191, 10000900001.

(2) The case of $30_n \times 0_n3$.

30103,
30203, 3002003, 300020003, 3000002000003, $30_6 20_6 3$,
30403, 3000004000003,
353, 30000500003, $30_8 50_8 3$, $30_{10}50_{10}3$,
373. 30703, 3007003, $30_6 70_6 3$,
383, 30803, 300080003, $30_7 80_7 3$,

(3) The case of $70_n \times 0_n7$.

727, 70207, 700020007,
$70_9 30_9 7$,
70507, 7000005000007,
70607, 700060007, $70_9 60_9 7$,
787,
797, 700090007, 7000009000007.

(4) The case of $90_n \times 0_n9$.

900010009,
9002009, 90000200009,
900050009, 9000005000009,
90709, 9000007000009.
$90_9 80_9$.

Note. Throughout the calculation, we saw that
90000900000909999099991
and its reverse, 19999099990900000900009, both are prime.

फार्म—४

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CONTENTS

Biology

1 **J. C. GHILDIYAL :** Effects of *Alstonia Scholaris* R. BR. Extracts on Seed Germination and Early Seedling Growth of *Parthenium Hysterophorus* Linn.

7 **A. K. AGRAWAL & S. K. GUPTA :** Growth Rate, Production and Energy Accumulation Efficieny of *Dendrocalamus Strictus* Nees. (Bamboo) in Western Himalaya, India

Chemistry

15 **S. KAUR & G. K. SINHA :** Chemical Investigation of Some Indigenous Basil Oils

Physics

23 **R. P. SINGH & S. L. SINGH :** Electrodiffusion of Silver in Zinc Phosphide

31 **S. P. SINGH & M. C. JOSHI :** Model of Lattice Dynamics for HCP Metals

39 **A. P. PATHAK :** Ion Implantation and Defect Studies in Semiconductors

Mathematics

53 **M. STOJAKOVIC :** Coincidence Point Theorems for Multivalued Mappings in Menger Spaces

63 **S. A. NAIMPALLY & G. DI MAIO :** Proximity Extensions

67 **K. ISEKI :** Determination of Special Palindromic Prime Numbers

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol 3 (1989). 1-5

EFFECT OF TEMPERATURE ON DEVELOPMENT OF CYCAS OVULE ROT AT HARDWAR

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ABSTRACT

Present paper deals with the effect of temperature on incidence of Cycas ovule rot at Hardwar. Pathogenecity experiment of Cycas ovule rot due to *Fusarium moniliforme* and *F. moniliforme* var. *intermedium* was conducted at 30°C and 25°C temperature respectively. Larger sized and dark brown coloured lesions were produced by both pathogens at 30°C but lesions were of smaller size and of light brown colour at 25°C. Maximum growth of both pathogens was found at 30°C which was 430 mg (DW), 8.6 cm (diametre of colony) & 570 mg (DW), 7.6 cm. respectively.

Key Words : Cycas, Ovule Symptoms, growth, Pathogen.

INTRODUCTION

Cycas a well known ornamental plant, includes twenty species and grows from Japan to Australia and Madagoskar (Africa). It is a slow growing long lived and evergreen palm like plant. Six species of *Cycas* are found in our country and *C. revoluta* Thenub is the commonest species. Generally female plants of this species are found only in northern India.

The incidence of the disease was observed in the month of October 1986 on the ovules of *C. revoluta* grown in the Botanical garden of

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Gurukul Kangri University Hardwar. Persistence of infection on ovule has been noted upto month of March in year 1987, 88 and 89 regularly. About sixty to seventy percent ovules were infected with the disease. Dry ovule rot of Cycas was a new disease in this country and the pathogen involved in the disease has not been reported from country till now (1,2,4,5). The pathogens were identified as *Fusarium moniliforme* and *Fusarium moniliforme* var. *intermedium* by CMI, Ferryland, Kew surrey, England.

As temperature and humidity are very important factors in development of any disease and growth of the pathogen therefore present investigation was undertaken to find out the relationship of temperature with incidence of disease and growth of the pathogen.

MATERIALS AND METHODS

Medium sized ovules showing typical characteristics of the disease (dark brown lesion on ovule surface), were collected from the plants and were brought to the laboratory for isolation of the pathogen and further studies. Ovules were surface sterilized by treating with 0.01% $HgCl_2$ solution for 15 seconds to kill the surface microflora. Treated ovules were repeatedly washed 5 to 6 times with sterilized water. These ovules were cut into small pieces with the help of sterilized blade and were inoculated on PDA plates. Certain ovule pieces were placed on PDA with their inner surface. Inoculated petridishes were incubated in BOD incubater at 26±1°C for growth of the pathogen. The pathogens were purified and stocked on PDA slants. Cultures were sent to CMI and were identified as *Fusarium moniliforme* and *Fusarium moniliforme* var. *intermedium*.

These cultures were used for pathogenecity experiment invitro at 25°C and 30°C temperature. Growth of the pathogens (radial and dry weight) was also studied at above mentioned temperature to see the effect of temperature on development of disease and growth of the pathogens.

Pathogenecity Experiments in Vitro at Different Temperatures

Medium sized healthy ovules showing typical characteristic orange colour were used in the present investigation. The ovules along with megasporophylls were surface sterilized with $HgCl_2$ solution of. 01% for twenty seconds and washed repeatedly with sterilized water. Injuries were made on the ovule surface and injured ovules were inoculated with discs of 8mm. of both pathogens. Discs were taken from edges of six days old colony grown on PDA medium. These inoculated ovules were placed in sterilized enamel trays and were covered with glass beljar. Sterilized wet cotton was placed inside the beljar to provide hundred percent humidity for development of the disease symptoms. Inoculated ovules with both pathogens were incubated at 25°C and 30°C temperature respectively in BOD incubator for development of disease symptoms. Observation was made after 10 days of incubation.

Growth of Pathogens at Different Temperature

The pathogens were grown on potato dextrose liquid broth (PDB) and on PDA medium to find out the effect of temperature on growth of the pathogens. Earlier plated Petridishes were inoculated with the discs of 8 mm. diameter of both pathogens and were incubated in BOD incubatore at 25°C and 30°C for their radial and mycelial growth. The flasks of 250 ml. capacity containing 30 ml. sterilized PDB were inoculated with 8 mm. discs of the pathogens and were incubated at 25°C and 30°C temperature. Diameter of the colony and dry weight of the mycelium was recorded after six days of incubation.

Result and discussion

Temperature has shown profound effect on development of disease symptoms (lesions) on Cycas ovule. It has been found that lesions were much clear, dark brown coloured and larger sized at 30°C than at 25°C in both pathogens i.e. *F. moniliforme* and *F. moniliforme* var. *intermedium* respectively (Table-1).

Maximum diameter of colony as well as dry weight of mycelium in both of pathogens occured at 30°C temperature. Record of dry weight of mycelium showed profound effect of temperature on growth of pathogens (table 2).

Higher growth rate of both pathogens at 30°C is directly related with the incidence of the disease. As both pathogens grew well at 30°C temperature it indicates that pathogens could increase the inoculum potential at 30°C which is an important factor for development of any disease. Day temperature of Hardwar during October and November months was found 30°C in the garden and campus. Symptoms of the disease on Cycas ovule was also observed during October 1986. Luxurient growth of both pathogens at 30°C temperature invitro and incidence of the disease on Cycas ovule at this temperature showed positive correlation in between temperature and disease development. Kulkarni (3) also studied the temperature effects on development of cotton wilt caused by *F. oxysporum, F. vasinfectum* (Atk) Snyder and Hansen and found that 31°C temperature was maximum temperature for disease development.

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TABLE-I

Characterestics of 6ycas ovule rot at different Temperature

Organisms	Incubation Period	Temperature 0°C	Average area of Lesion in cm.	Characterestics of ovule rot
Fusarium moniliforme	10	25	1.00 to 1.5	Generally Circular Semicircular and light in colour
	10	30	2.0 to 3.3	Circular and dark brown colour
F. moniliforme-var. -	10	25	1.0 to 3.3	Circular and dark brown colour.
intermedium	10	30	2.0 to 4.00	Circula and dark brown colours.

TABLE II

Influence of Temperature on growth of *F. moniliforme* and *F. moniliforme* var. *intermedium*

Organisms	Temperature (°C)	Incubation days	Average Diameter of colony in cm.	Dry weight of mycelium in mg.
F. moniliforme	25	6	8.0	400.0
	30	6	8.66	430.00
F. moniliforme var. intermedium	25	6	6.6	552.5
	30	6	7.91	570.0

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (89) 6-11

EFFECT OF INDUSTRIAL EFFLUENT ON SEED GERMINATION OF SOME AGRICULTURAL CROPS

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ABSTRACT

The present investigation was conducted to find out the effect of industrial effluent of two industries viz. Doon distillery and Indian Drug & Pharmaceutical Limited on seed germination of four agricultural crops i.e. *Triticum vulgare, Luffa cylindrica, Abelmoschus esculentus* and *Phaseolus mungo*. During this study, maximum percentage of seed germination was recorded in wheat seed i.e. 86.6 at 100 per cent concentration of IDPL effluent (E_2). A gradual fall in percentage of seed germination of wheat crop was recorded with dilution of this effluent. Adverse effect of Doon Distillery effluent (E_1) was recorded in seed germination of the all test crops as compared with control.

Key words :– Pollution, Effluent, Germination, percentage.

INTRODUCTION

Water pollution is a major problem of the world. The problem appears quite serious specially for the surface water which receives various kinds of pollutants through various means. It can be stated truly that various larger and smaller stream and other aquatic bodies in the country are in danger. River Ganga is also facing the same problem. It receives various kinds of pollutants in various form i.e. industrial effluent, sewage and other various kinds of garbage throughout the course.

In Rishikesh and Hardwar region, various industries such as IDPL located at Virbhadra, Rishikesh and BHEL may be considered as big source of industrial effluent. Discharge of the effluent of these units into Ganga resulted drastic change in water quality (Shanker [8]. Doon Distillery located at KUANWALA, District Dehradun discharges large quantity of its effluent into Reh. This effluent has indirect effect on water quality of Ganga through river song.

The effect of various kinds of pollutants on water quality in different aquatic system has been studied in the country by various workers (see for instance, [2] and [8]. Albeit the use of sewage for irrigation in agricultural crops is in the practice, but industrial effluent is still discharged either on the earth surface or in aquatic system. Discharge of industrial effluent directly in the streams, not only deteriorates the water quality but has adverse effect on aquatic flora and fauna. Shanker *et al.* [9].

The present investigation was undertaken to find out possible utilization of industrial effluent for irrigation. A large area of agricultural land is found in surrounding areas of IDPL and Doon Distillery, and after positive findings, effluent may be used for irrigation purposes so that the effluents are put to economic use in terms of providing irrigation water to fertilizer.

MATERIALS AND METHODS

Industrial effluents of Doon Distillery and IDPL were collected during 1985 at an interval of one month and were evaluated according to APHA for their certain physico-chemical characters (see) [1]. The effluents were further used for the treatment of seeds of test crops. Four different kinds of seeds i.e. *Triticum vulgare* var. RR-21-, *Luffa cylindrica* var. Faizabadi, *Abelmoschus esculentus* and *Phaseolus mungo* were used as test organisms.

Seeds of the above test crops were surface sterilized with. 1% $HgCl_2$ solution and were repeatedly washed with sterilized distilled water for 6-7 times. After proper washings, these seeds were treated in three different concentrations of each effluent i.e. 100 per cent (Pure effluent), 50 per cent and 25 per cent. Three replicates were used for each concentration. Seeds as such (without treatment) were used for control. Treated seeds were kept in moist chamber, prepared in petridishes of 10 cm diameter with blotting sheets and effluent. Ten seeds were kept in each petridish. These petridishes were kept in room temperature for germination of seed. Percentage of seed germination was recorded after 4 days. Investigation was reconducted three times to confirm the results.

RESULTS AND DISCUSSION

Results of the present investigation have been given in Tabels 1 and 2. Average value of physico-chemical characters of the effluent for the period of a year i.e. 1985 has been given in Table 1. Results revealed that effluent of both industries is highly polluted and contains higher level of total solids, BOD and COD etc. beyond permissible limit (Table 1). Dissolved oxygen was very poor in IDPL effluent and was found always nil in Distillery effluent. Similarly, very low level of pH was recorded in the distillery effluent.

Findings on effect of effluent on seed germination showed that IDPL effluent (E_2) has promotive effect as percentage of seed germination in general except *Phaseolus mungo*. (Table 2). Maximum percentage of seed germination among all the test crops has been recorded in *T. vulgare* i.e. 86.6 at 100 per cent concentration (without dilution). A gradual fall in percentage of seed germination was recorded with dilution of the effluent at 50 and 25 per cent (Table 2). These findings indicate that IDPL effluent contains some nutrients and fulfill the requirement of manures to the germinating and growing crop up to certain extent. Furthermore, the results are encouraging in case of *L. cylindrica* and *A. esculentus* where seed germination was recorded slightly better than the control.

Adverse effect of Doon Distillery effluent (E_1) was recorded on seed germination in all the test crops as compared with control. There was no seed germination at 100% (pure effluent) in *T. vulgare* and *L. cylindrica*, while very poor seed germination was recorded in *P. mungo* and *A. esculentus* (Table 2). Enhancement in percentage of seed germination was found in *A. esculentus* with dilution of effluent at 50 and 25 per cent

(Table 2). However, this did not show any encouragement in case of *T. vulgare* (Table 2).

The poor germination with slight increase with dilution of effluent showed the dilution of various toxic substances which are present in dissolved and suspended particles, BOD and COD etc. as well as this accelerate of pH level of effluent (E_1) i.e. 3.3. Higher concentration of total solids i.e. 25180 mg/lt may be a factor for inhibition of seed germination. Similar finding has been found in seed germination of *Hordium vulgare* by Bahadur and Sharma [3]. Similarly, Sahai and Saxena [7] have also reported harmful effect of effluent on seed germination. Absence of oxygen may also be a cause of inhibition of seed germination in distillery effluent (E_1) as reported by Bahadur and Sharma [3] in *H. vulgare*. Adverse effect of this effluent (E_1) may also be due to another factor i.e. very low level of pH 3.3. As seeds in control were shown in tap water and in general pH of water as found in range 6.8-7.0. The inhibition of seed germination may be due to combined factor rather than any single factor.

Summarily, effluent of IDPL factory (E_2) may be recommended for irrigation for *T. vulgare*., *L. cylindrica* and *A. esculentus* crops specially for wheat but before to recommend it is highly needed that aforesaid crops should be evaluated for the plant health, yield, fodder quality as well as food and vegetable quality. A biochemical study of the plant and its product is also necessary so that if there is any kind of toxic substances are stored in plant or its product can be estimated quantitatively as well as qualitatively. Albeit in priliminary stage, results of IDPL effluent are promising but cannot be recommended for irrigation till the above information is *not* received experimentally.

TABLE–1

Annual average value of Physico-chemical characteristics of Effluents (1985)

Effluent	pH	Total solids (mg/lt)	D.O. (mg/lt)	BOD (mg/lt)	COD (mg/lt)
E_1	3.3	25180	Nil	7252	15700
E_2	6.9	1200	2.85	189.5	24000

E_1 = Distillery effluent

E_2 = IDPL effluent

TABLE–2

Effect of Effluent on Seed germination of some agricultural crops

Effluent	Treat-ment (%)	Test crop & Percentage of Germination			
		T. *vulgare*	L. *cylindrica*	A. *esculentus*	P. *mungo*
	100	Nil	Nil	20	0.5
E_1	50	Nil	10	55	17.5
	25	10	43	55	17.5
	100	86.6	56.2	65	Nil
E_2	50	70	59.2	67	Nil
	25	70	36	67	--
Control		53	60	68	75.5

E_1 = Distillery effluent

E_2 = IDPL effluent.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (1989), 12-32

THEORY OF OLFACTION/GUSTATION BASED ON DIPOLE MOMENT OF MOLECULE

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ABSTRACT

Various Investigators have proposed many theories to explain the phenomena of olfaction [1, 19] and gustation [16], but most of them touched the boundary of the problem. The theory presented in this paper could explain all the aspects of the two phenomena.

The theory helps in understanding the smell and taste (Chemical) senses completley just like there is complete understanding of light and sound senses.

1.00 Introduction : Eyes, ears, nose, tongue and skin are the parts through which sensations are felt by human body. Light and sound sensations felt through eyes and ears have been completely explained by the scientists. However smell and taste sensations felt by the nose and tongue respectively could not be explained satisfactorily by various theories proposed till now. The most popular theory of olfaction has been proposed by Amoore et al. (1) which relates odour quality with shape and size of molecule. Shallenberger et al. [16] proposed a theory of gustation co-relating sweet taste with structure of molecules.

The theory presented in this paper explains completely the two phenomena of chemical senses.

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2.00 Receptors

2.01 Olfactory receptors : There are two types of olfactory receptors; Epithelial and Antennae type. Epithelial type receptors are found in animals, birds, reptiles, fishes etc. and are embedded and flushed with skin surface, usually in their nasal cavity. The general features are similar in them (15). The antennae type receptors are found in insects.

Consider the human olfactory cell as a particular case. The olfactory cell mainly has olfactory bipolar neurone from which cilia emerges. The secretions of Bowmann's glands keep the surface of epithelium moist and furnishes the necessary solvent. The odour molecule reaches the sensitive bipolar neurone after being dissolved in these secretions.. The continuous stream of secretions removes the remains of the stimulating substances and keeps the receptors ready for new odour molecules.

The general features of antennae type receptors of insects are similar. It contains naked dendrites bathed in biological fluid and are surrounded by cuticle having pores. The odour molecules reach sensitive dendritic membrane through these pores, after being dissolved in biological fluid (10).

2.02 Gustatory Receptors : These are groups of taste buds distributed usually on tongue. Each taste bud is a group of cells arranged in a small well around nerve endings. The molecules to be tasted reach these nerve endings by dissolving in water or saliva.

The above facts show that it is the nerve endings which receive odour or taste molecule (hereafter called guest molecule) dissolved in biological fluid. After the molecule has reached nerve endings the phenomena of odour and taste are indentical and further discussions apply equally to both these phenomena.

In case of taste phenomenon, it has been shown that it is not dependent on temperature in the range of 20°C to 30°C (near body temperature), [2] and the same is expected in case of odour phenomenon.

3.00 Molecule and nerve membrane : The guest molecule from air moveş to biological fluid medium in which the nerve endings are bathed and approaches towards the membrane of nerve endings (2.02). The condition of molecule in biological fluid is as described below.

3.01 Structural model of molecule in biological fluid: Since the biological fluids mainly contain water (nearly 80%), [9] a structural model must be formulated on the basis of structure of molecule in water.

The most accepted water hydrate model is being adopted [4]. The water molecules in liquid water exist in monomeric and polymeric forms, forming ice-like structure having cavities, which are capable of accommodating guest molecules. Two types of cavities of sizes 12.03 Å and 17.31 Å diameter have been suggested. The water molecules are hydrogen bonded with each other and have no free electron or proton to make further hydrogen bond. Dimers and trimers are very common.

Most of the gas hydrates constitute a class in which small molecules of many types occupy almost spherical holes in ice-like lattices made up of hydrogen bonded water molecules. The guest molecule will form such types of gas hydrates in water/biological fluid when dissolved in it. The guest molecle remains encaged in the firm enclosure made up of these H-bonded water molecules. The order of water molecules remains same. The guest and water molecules are not held up by any strong force and there is a remarkable degree of rotational mobility on the part of encaged molecule behaving as if in a vapour state [7]. The interaction of water and guest molecule appears primarily of van der Wall's type even for polar guest molecule. Guest molecule upto 5 Å in size fit easily into cages while a larger molecule, which can not be accommodated in a single cavity may occupy several adjacent sites.

The above model has been adopted for the interaction of guest molecule with nerve membrane. Before discussion of the interaction, some points regarding membrane and its excitation are discussed in the following para.

4.00 Membrane and excitation:

4.01 Membrane protein : It has been established that in the membrane, the lipid bilayer forms a barrier while its protein molecules after suffering conformational change are responsible for excitation. The exact mechanism, of excitation has not been understood. Thus, it is concluded that the guest molecule present at the outer side of nerve membrane, interacts with its protein, causing conformational change in it, resulting the excitation [11].

The globular protein of membrane contains 30% to 100% α-helix besides other parts. There is no water inside α-helix of protein. It has a delicately balanced structure. It performs highly specific and complicated chemical actions, which require high degree of organisation. In living organism it exists in native state. We have very little knowledge about its native or denatured state. The incorporation of certain proteins and polypeptides in artificial lipid bilayer may acquire selectivity, electrical excitability and even chemical receptor properties.

Protein molecules have two types of bonds; strong (Co-valent) and weak (Non-covalent) bonds. The weak hydrogen bonds are the predominent bonds governing folding and unfolding of protein.

Since no monopoles are observed inside the protein, the interactions of the multipoles are short and may be restricted to close neighbours only, in case of energy calculations.

Therefore, if the native protein is under stress, which increases continuously from zero stress (native state) to a higher value, the weak hydrogen bonds will break first resulting in its denatured state. This is a important conclusion for the present studies. Breaking of H-bond in protein of retina has been taken as primary cause of vision perception (14).

4.02 Model of membrane protein : A model of protein molecule has to be built up for further studies which should fulfil the excitation phenomenon. The main features of the proposed model are :

(a) Membrane excitation is due to conformational changes in the protein molecule.

(b) Structure, formation and stability of protein molecule depend mainly on non-covalent bondings.

(c) Hydrogen bonds are most important non-covalent bonds responsible for conversion of protein from native to denatured state which results in excitation in membrane.

Major portion of protein molecule comprises of right handed α-helix. It has been speculated that a reversible transition (from α-helix to sheet structure) may be the cause of muscle contraction and other types of movement in living organism. In a number of enzymes the biological activity depends on α-helical content (5). Thus it is concluded that :

(a) The globular protein is a delicately balanced structure and a small change in its energy may lead to a large change in its overall structure.

(b) Breaking of H-bond of α-helix, leading to conformational changes in protein molecule is the most probable cause of excitation.

(c) α-helix are present almost always on the surface of the protein [3].

On the basis of above facts, the following model has been adopted for further studies. In this model the polypeptide chain of protein is folded into α-helix. Th folding is maintained by H-bonds between -NH and =CO groups. Helix are assumed to be held rigidly with each other by di-sulphide bonds at some places (Fig. 1).

5.00 Interaction of guest molecule with protein : The Dipole Moment (D.M.) :

Interaction of guest molecule with protein molecule results in biological activity. Important facts regarding the interaction are as below:

(i) Guest molecules, under consideration may be taken as small molecule which neither react with water nor ionise in it e.g. O_2 CH_3, OH, $C_4 H_9 NH_2$ etc.

(ii) Protein molecule has α-helical content as a major portion bathed in water medium. It is inpenetrable to solvent and is surrounded by a firm layer of water of uni-molecular thickness [9].

(iii) The guest molecule when dissolved in water, lies in the interstitial cavity of water medium and forms a hydrate with it. The concentration of guest molecue is infinitly low ($< 10^{-3}$ M), [18] so that it can be considered as an isolated molecule, not affected by the presence of other similar molecules.

(iv) In a state of infinite dilution, the structure of guest molecule does not change and can be compared with its structure in the gaseous state [4]. There is very weak coupling between guest molecule and surrounding water molecules. At low concentrations any marked effects normally represent specific interactons of co-solute (guest molecule) with protein [13].

(v) The protein molecule consists of various bonds out of which H-bonds are weakest. Any small strain on these H-bonds may result in a major change in the protein molecule because it has a delicately balanced structure. Since denaturation process has been split into two step transition. Firstly native to a weekly co-operative transition and then to a state of denaturation (6).

We are interested in the strain produced on H-bond due to the presence of guest molecule which may result into complete danaturation process or a part of it as suggested above.

Other reasons for consideration of H-bonds of α-helix are:

1. α-helix are most abundent, delicately balanced structure with H-bonds continuously breaking and forming. They are found at the surface of protein.

2. Transition of α-helix to β-configuration may be responsible for biological activity which is the main consideration.

(vi) Now suppose a guest molecule 'M' when dissolved in water (or biological fluid), is approaching the helical part of protein molecule. When it is at the closest state to protein molecule, the situation is as shown in fig. 1. This type of bonding is reversible (8).

(vii) In the situation as depicted in para (vi), the stability of a H-bond due to its interaction with guest molecule 'M' (Say CH_3 OH) is to be investigated. A further Simplification gives the pictute (fig. 2) with only one H-bond and part of guest molecule in adjacent cavity. Here interactions except guest molecule and H-bond are neglected (17).

(viii) When there is no guest molecule 'M', the H-bond is stable with cage structure of water medium not occupied. The only difference between this situation and the situation depicted in Fig. 2 is that the cage structure in former is not occupied by a guest molecule, while in later case it is occupied.

(ix) There are two types of cages having exterior sizes equal to 12.03 Å and 17.31 Å giving an average exterior size as 14.68 Å. The average internal size is 4.2 Å.Thus the average thickness of cage is 3.14 Å.Take the distance of H-bond from unimolecular water layer as 1 Å. In case of polar guest molecule its polar part is most likely to be nearer to H-bond (Fig. 2 for CH_3 OH Molecule) and its minimum distance from H-bond will be little more than $1.52 + 3.14+1.0 = 5.66$ Å Say 6 Å. Thus the dipolar part of guest molecule can not come closer to H-bond than this value. If the molecule is bigger and can not be accommodated in one cavity, its polar part will lie nearer to H-bond and non-polar part will occupy adjacent cavities.

(x) When 'M' has entered the hydrate cage, it will interact electrostatically with H-bond.

(xi) The interaction energy between 'M' and H-bond is to be estimated. This is an additional energy due to presence of 'M' and will strain H-bond. If the interaction is of attractive in nature, the energy is surplus.

(xii) The interactions are due to di-or multipoles and therefore are short range interactions. This may be restricted to close neighbours only.

(xiii) The approximate maximum surplus strain energy experienced by H-bond due to presence of 'M' (Fig. 2) can be computed by Stockmeyer's potential. This energy,

$$E = 4\,\varepsilon\left[\left(\frac{\sigma}{\gamma}\right)^{6} - \left(\frac{\sigma}{\gamma}\right)^{12}\right] - \frac{2\,\mu_1\,\mu_2}{\gamma^3} \quad (\rightarrow\rightarrow)$$

(xiv) If the distance of separation 'r' between 'M' and H-bond is medium (6 Å to 15 Å) the 1st part of the equation (i) is quite low as compared to IInd part and energy E can be approximated to:

$E = \frac{2\mu_1\,\mu_2}{D^{*} r^{3}}$, where 'D*' is the dielectric constant.

(xv) D.M. of H-bond 'u' is a fixed quantity, thus

$$E \,\alpha\, \frac{\mu_2}{D^{*} r^{3}}$$

(xvi) If the protein molecule and medium remain same (as in case of nerve membrane) and value of 'r' is constant (in case of closest approach of polar part of 'M' to H-bond), the energy is dependent on 'μ_2' i..e. D.M. of guest molecule only.

(xvii) This (excess) energy will strain H-bond and may break it, if it is more than the bond energy of H-bond.

(xviii) Any strain to H-bond may result in drastic changes to the structure of α-helix which has a delicatly balanced structure. The Co-valent bonds of α-helix may act as an amplifier in magnifying the small changes in its structure into re-organisation of the molecule on a larger scale.

(xix) The dielectric constant 'D*', has been assumed to vary with 'r' in a linear manner with values as below :

r	o Å	6 Å	30 Å
D*	1	2	80

5.01 Hydrogen bond and interaction energy : Various investigators have given the following values of H-bond energy ; 0-2 K cal/mole, 100 cal/mole. A lower value of the bond energy has been suggested in aquous medium [6]. A value of 100 cal/mole seems to be a reasonable estimate.

As a typical case, interaction energy has been calculated for the case of interaction of CH_3 OH molecule with H-bond. The values of parameters are :-

CH_3 OH (molecule)		H-bond
value of 'ε'	0.82 K cal/mole	0.073 K cal/mole
value of 'σ "	3.69 Å	3.0 Å
value of 'μ'	1.7 D	1.3 D

Taking value of r=6 Å and dielectric constant D*=2, the total energy is nearly.90 cal/mole. The dipole-dipole interaction energy is 84% of total energy and this % increases with increase in value of 'r'.

Thus it is seen that the interaction energy is of the order of H-bond energy. It should also be kept in mind that the nature performs its functions in a more efficient manner and may need lesser energy than what has been estimated (see also para 5.00 v).

5.02 Conclusions : It has been seen that the guest molecule is capable of straining (or breaking) H-bonds, which brings a major change in protein of dendritic membrane, resulting in biological activity. The strain caused to H-bond is proportional to the Dipole Moment of guest molecule only (see para 5.00). Thus it is concluded that the Dipole Moment of guest molecule may be related directly to the biological activity of dendritic membrane of olfactory/gustatory receptors causing typical odour/taste.

With these ideas a plot of molecular Dipole Moment and its characteristic odour and taste (called odour/taste spectrum) has been proposed to check the validity of above conclusion.

6.00 THE SPECTRUM

6.01 Description of Odour and Taste: Odour of many molecules have been described by investigators in a variety of ways and it is little difficult to pin-point the correct statement. Therefore, a personal judgement has been applied in such cases in assigning a typical odour for these molecules. However, many molecules have been left out of this study where the odour description has not been precise e.g. 'sharp penetrating', 'mild odour', 'pungent' etc. Only those molecules have been selected where their odours have been described as typical odour e.g. 'rose odour', 'pear odour' etc.

Description of taste of molecules is even lesser in number and less descriptive. The procedure followed in case of odour molecule has been adopted in selecting the taste of a molecule.

6.02 Intensity : It has been felt that intensity of odour/taste depends upon the rate of molecules reaching the membrane of receptors, which depends upon the physical parameters such as wind velocity, temperature, rate of out flow of molecules from the source, its water solubility etc. It has been observed that odour/taste of a molecule changes due to wide variations in its intensity. This may be due to some different changes in membrane protein because of presence of molecules in higher number than what is considered here i.e. in infinitly low number. In present studies this factor of higher intensity has not been considered.

6.03 Odour/Taste Molecule: Most of the molecules under consideration exist in monomeric forms e.g. halo-forms, highly halogenated compounds, acytylenes, ketones, ethers, esters, olefines, aromatic and tertiary amines, nitriles, isonitriles, saturated hydrocarbon, carbon di-sulfide etc.

There are molecules which self associate to give a bigger molecule e.g. phenols, alcohols, inorganic and carboxylic acids, primary and secondary amines etc. Even these molecules exist in monomeric forms if concentration is less than 10^{-3} M.

The molecules falling in above categories have been considered in this study. Molecules reacting in atmosphere or reactng with membrane components have been excluded from these studies.

6.04 Value of Dipole Moment : Due to following reasons difficulty has been felt in selecting the value of D.M. of a particular molecule.

(a) These values have been reported for gas form as well as in solutions of various solvents at different temperatures. The solvent effect of non-polar solute has been found to be negligible (12). However, gas values near body temperature has been selected as most appropriate value.

(b) In many cases very few values are available and it is difficult to assess the correctness of the reported value (s). Such values have been used with some restraint.

6.05 Odour/Taste Spectrum: On the basis of conclusion drawn in para 5.02 that the **molecular dipole moment is directly related to its characteristic odour/taste;** a spectrum of Dipole Moment expressed in Debye unit 'D' and the characteristic odour/taste of the molecules has been proposed (See Annexure). Its general features are:

(i) The plot ranges from D.M. = 0 to 3.86 D

(ii) The spectrum is applicable in case of odour/taste molecules present in low concentrations.

(iii) There are 113 distinct odours and 48 distinct tastes.

(iv) Molecules with D.M. < 0.15 D are odour/tasteless. This indicates that the interaction energy in such cases is not enough to cause excitation to the extent that it is perceptible to human organs.

(v) The spectrum is broadly divided into following parts.

D.M. = 0 to 0.15 D-odour/tasteless
= 0.15 D to 0.4 D-Hydrocarbon odour/taste
= 0.4 D to 0.65 D-Oily odour/taste
= 0.65 D to 1.75 D-Mixed odour/taste
= 1.75 D to 1.95 D-Fruity, Flowery odour/taste
(values close to water)
= 1.95 D to 2.40 D-Animal odour/taste
= 2.40 D to 3.56 D-Perfumery odour/taste.

(vi) Water occupies a special status. It is odour/tasteless even though its D.M. value is very high. It is because when water molecule reaches near membrane which is already bathed in water medium, the situation near it does not change at all. Therefore, water is odour/tasteless.

7.00 Conclusions : The various conclusions drawn from present studies are listed below:

(i) The excitation in olfactory/gustatory organ is due to interaction of odour/taste molecule with its sensitive membrane.

(ii) Molecules having same D.M. have similar odour/taste, while molecules of different D.M. have different odour/taste. This verifies the inference drawn in para 5.02 that D.M. is directly related to the biological activity of odour/taste molecules.

(iii) Same value of D.M. is representative of both odour and taste of the molecule simultaneously e.g. D.M. = 1.44 D corresponds to acetic acid odour as well as acetic acid taste.

(iv) The molecules of a group of hydrocarbon having same polar part but different chain length have similar odour/taste. It is because the polar part is nearer to H-bond of membrane protein while the tail is away and the effect of tail is very less as compared to its polar part.

(v) The resolution power of human (Olfactory/Gustatory) receptor membrane in terms of D.M. is 0.01 D i.e. the membrane in both cases is capable of distinguishing molecules if their D.M. differ by more than 0.01 D.

(vi) The general conclusions drawn from studies of human receptors seems to apply in case of receptors of other vertebrates, insects, birds etc.. because there seems to be no reasons contradictory to it in light of para 2.00. Here it is emphasized that human olfactory sense is quite inferior to many animals and insects.

(vii) Pleasent or unpleasentness depends upon individual nature. Normally all odour/taste, felt at low concentrations are pleasent and are unpleasent at high concentrations.

(viii) The spectrum will help in predicting odour and taste of a molecule of known value of D.M. and vice-versa. Knowledge of D.M. will help in finalising molecular structure of a molecule.

(ix) The spectrum will be helpful in finding out new synthetic flavours.

ANNEXURF

Odour/Taste Spectrum O = Odour

T = Taste

D.M. (D)	Odour/Taste	D.M. (D)	Odour/taste.
0.00	Odourless/Tasteless	0.85	Iodoform (O)
0.10	Odourless/Tasteless	0.87	Chloral (O)
0.15	Nitrous Oxide (O)	0.90	Rotten egg (O)
0.20		0.95	Chloroform (O, T)
0.25		1.00	Grapes (O, T)
0.30		1.01	Orange blossom (O)
0.35	Hydrocarbon (O)	1.02	
0.40	Kerosine (O)	1.03	Metallic (O)
0.45	Tarpene (O)	1.04	
0.475	Benzene (O)	1.05	Gerfanium (O)
0.50	Fried Potato (O..T.)	1.06	1.68
0.55		1.065	Napthalene (O)
0.56	Ozone (O)	1.08	Fishy (O)
0.60		1.10	Halogen (O)
0.65		1.11	
0.70		1.12	
0.75		1.13	Chloroethynyl Benzene (O)
0.80		1.14	

1.15		1.46	
1.16	Diethyl ether (O)	1.48	
1.17	Sweet (T)	1.49	Ammonia (O)
1.18	1.805	1.50	
1.19	Caproic (O, T)	1.51	Cabbage/radish (O,T)
1.20		1.52	
1.21		1.53	Sulfuraceous/Shunk (O)
1.22		1.54	
1.23	Piperidine (O)	1.55	Thyme (O)
1.24		1.56	
1.25	Anisole (O)	1.565	Peppery (O)
1.26		1.575	Woody (O, T)
1.28		1.58	
1.30	Marigold/wall flower (O)	1.585	Dry tea leaves (O)
1.32		1.595	Ylang-Ylang (O)
1.34	Hyacinth (O)	1.60	Narcissus (O)
1.36	Earthy (O)	1.605	Lily (O)
1.37	Hazelnut (O, T)	1.61	Jasmine (O)
1.38		1.62	Orrissy (O)
1.40	Rancid Buttery Cheesy (O, T)	1.64	Dry leather (O)
1.42		1.66	Phenolic/medicinal (O, T)
1.44	Acetic acid (O,T)	1.67	Chemical (O, T)

1.685	Fusal oil (O)	1.89	Anisic (O)
1.70	Alcohlic (O, T)	1.88	Strawberry (O, T)
1.72	Mandrain (Citrusy) (O, T)		1.90
1.73	Lemon (O, T)	1.91	Oily burnt (O.T.)
1.74	Orange (O)	1.92	Rosemerry/Lavandin
1.76	Waxy (O)	1.93	Piney (O)
1.77	Honey (O, T.)	1.94	
1.78	Rose (O, T)	1.96	
1.79		1.98	
1.80	Apple (O,T)	1.99	Onion (O.T.)
	Pear (O, T)		
1.81	Banana (O, T)	2.00	Garlic (O.T.)
1.815	Pineapple (O, T)	2.02	
1.82	Apricot (O, T)	2.04	
1.83	Blackcurrent (O, T)	2.06	
1.835	Water (odour & Tasteless) (O, T)	2.07	Orange leaf/flower (O)
1.84		2.08	
1.845	Ashtray (O)	2.12	
1.85		2.14	Animal (O, T)
1.86	Brandy (O)	2.155	Indolic (O)
1.87	Rum (O, T)	2.16	

2.17	Tarry (O)	2.50	
2.18		2.51	Wintergreen (O.T.)
2.19	Nitric acid (O)	2.52	
2.20		2.54	
2.21	Burnt/smoky (O, T)	2.55	Clove Oil (O.T.)
2.22		2.56	
2.24	Buttery (O, T)	2.58	Peach (O.T.)
2.25	Coconut (O, T)	2.60	
2.26		2.62	
2.28	Earthy (O)	2.63	Musty (O)
2.30	Oily (O)	2.64	
2.32	Walnuts (O)	2.65	Gassy (O)
2.34	Soapy (O)	2.66	
2.36		2.67	Tobacco leaves (O)
2.37	1, 5 di-chloropentane (O)	2.68	
2.38		2.70	
2.40		2.72	Roasted coffee (O)
2.42		2.74	
2.44		2.75	Bitter almond (O.T.)
2.45	Vanilla (O.T.)	2.76	
2.46	Chocolate (O)	2.78	Musky (O.T.)
2.475	Winey (O)	2.80	Mushroom (O)
2.48		2.82	

2.84			
2.85	Camphor (O)	3.16	Hawthorn (O)
2.86		3.18	
2.88	Peppermint (O.T.)	3.20	
2.90		3.22	
2.91	Acetone (O.T.)	3.24	
2.92		3.26	
2.94		3.28	
2.95	Balsamic (O)	3.30	Hay (O)
2.96		3.32	
2.98	Burning fat (O)	3.34	
3.00		3.36	
3.01		3.38	
3.02		3.40	
3.03		3.42	
3.04		3.44	
3.05	Mimosa (O)	3.45	Cederwood (O.T.)
3.06		3.46	
3.07		3.48	
3.08		3.50	
3.09		3.52	
3.10	New mown hay (O)	3.54	Heliotropine (O.T.)
3 12		3.56	

3.14		3.58	Benjoin (O)
3.60		3.80	Caramellic (O.T)
3.62		3.82	
3.64		3.84	
3.66	Bread (O.T.)	3.86	Orris/Ambre/vetiver (O)
3.68		3.88	
3.70	Cinnamon (O.T.)	3.90	
3.72		3.92	
3.74		3.94	
3.75	Spicy (O.T.)	3.96	
3.76		3.98	
3.78		4.00	

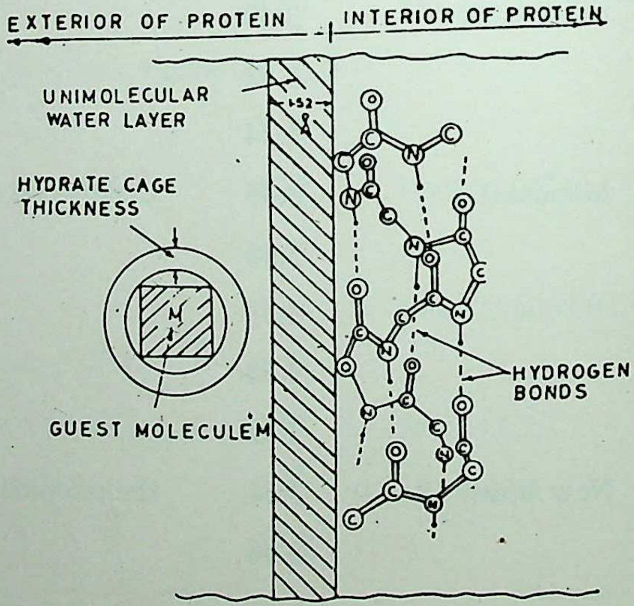
EXTERIOR OF PROTEIN
INTERIOR OF PROTEIN
UNIMOLECULAR WATER LAYER
1.52 Å
HYDRATE CAGE THICKNESS
M
GUEST MOLECULEM
HYDROGEN BONDS

FIG.-I HELIX AND GUEST MOLECULE

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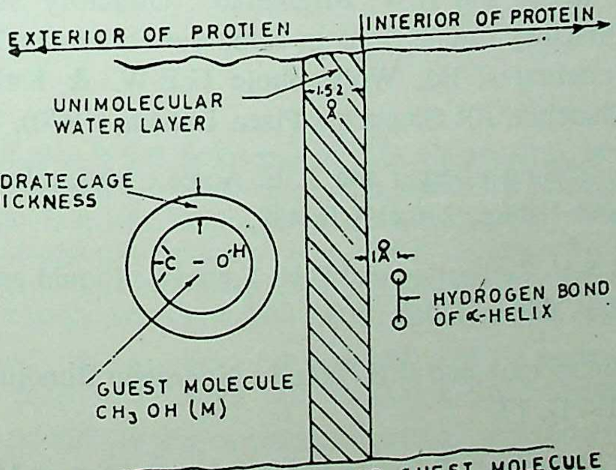
EXTERIOR OF PROTIEN
INTERIOR OF PROTEIN
UNIMOLECULAR WATER LAYER
1.52 Å
HYDRATE CAGE THICKNESS
C - O-H
1Å
HYDROGEN BOND OF α-HELIX
GUEST MOLECULE CH₃ OH (M)

FIG-2 HYDROGEN BOND AND GUEST MOLECULE

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A STUDY OF ESSENTIAL OIL FROM ERIGERON LINIFOLIUS WILLD

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ABSTRACT

Essential oil from *Erigeron linifolius* willd, a widely distributed plant in Kumaon and Garhwal regions, possessed

$$d^4_{20°} = 0.8782 \; ; \; n^D_{20°} = 1.53 \; ; \; \alpha^D_{20°} = 41° 21'$$

Acid value = 0.3 ; Ester value = 50.2 and Carbonyl value = 0.8. GLC study of the oil using Honeywell HP 7620–A model showed the presence of α-pinene, β-pinene, myrcene, ocimene, limonene, caryophyllene, humulene, cubebne and lachrophyllum ester.

Key words and phrases : Essential oil, *Erigeron linifolius Willd.*
Chemistry-subject classification : Pharmaceutical Analysis Essential oils-62.

Erigeron linifolius Willd, belonging to N.O. Compositae, is widely distributed in Kumaon and Garhwal regions. Leaves of the plant are used in Malaya for rheumatism, lumbago and poulticing [1]. Chemical composition of essential oils from *Erigeron annus*, *Erigeron karwinsky anus* DC, and *Erigeron philadiphicus* [2] [3] [4] have been reported but very little work is reported on essential oil of *Erigeron linifolius* Willd.

Experimental : The essential oil was obtained by water-steam distillation of the plant in a copper still, extracting the distillate with petroleum ether (b 60-80°), drying over sodium sulphate and concentrating the organic phase under reduced pressure (yield = 0.2%). Physico-chemical properties of the oil were :

Sp.gravity/$_{20}$°=0.8782 ; Ref. index/$_{20}$° = 1.53 ; Sp. rotation/$_{20}$° = + 41°21'

Acid value = 0.3 ; Ester value = 50.2 and Carbonyl value as $C_{10}H_{16}O$ (oximation) = 0.8.

Fine needle shaped crystals formed, on keeping the oil overnight during winters, were separated. The process was repeated till formation of crystals ceased. Hydrocarbon and Oxygenated fractions of the solid free oil were separated by using a column packed with silicic acid and n-hexane as eluting agents [5]. The two fractions were subjected to GLC examination, separately, under similar parameters. Different constituents of the fractions were identified by injection method and comparison of retention times under the same operating parameters using the pure compounds. The conditions of the GLC analysis were :

Column	..	Carbowax 20 M, 10%.
Carrier gas	..	Nitrogen (30ml per minute)
Column temperature	..	200°
Injection temperature	..	260°
Detector type	..	Thermal conductivity detector (temperature 300°)
Chart speed	..	0.25" per minute.
Sensitivity	..	10
Chromatograph model	..	Honeywell HP 7620-A.

Results of GLC of Hydrocarbon fraction (F1) and Oxygenated fraction (F2) are summarised in table 1 and table 2, respectively.

The solid fraction, separated from the oil earlier was further purified by column chromatography and TLC. Solvent systems used were chloroform, petroleum ether-pyridene (95:5). After development the plates were sprayed with fluorescein-bromine reagent for visual identification. Better resolution was obtained with petroleum ether pyridene. Two fractions, isolated, in pure form, were identified as cis-lachrophyllum ester and matricaria ester by Co-TLC and spectroscopic examination.

Spectroscopic examination of the fractions :

Fraction 1:

The fraction melted at 32°C and possessed molecular formula $C_{11}H_{10}O_2$ (C=75.8%, H=5.74% and 0=18.46%, Molecular weight = 174).

UV : λ max at 218 and 308 nm indicated the presence of symmetrical diene and three conjugated system units, respectively

IR : bands at 3650 cm^{-1} –3100 cm^{-1} (–CH–stretching); 2800 cm^{-1} 3000 cm^{-1} (CH_3–CH_2–$CH_2C{\equiv}C$–); 1600 cm^{-1} (strong) for carboxylate ion; 1750 cm^{-1} for α–β–unsaturated ester; 2200 cm^{-1} (medium) for –C ≡ C–; 970 cm^{-1} for –CH≡CH– and 1600 cm^{-1} for symmetrical diene.

IR spectrum of the compound was found the finger print of IR spectrum of matricaria ester.

Fraction 2 : The fraction melted at 33°C and possessed molecular forumula $C_{11}H_{12}O_2$ (C = 75.0%, H = 6.8% and 0 = 18.2%. Molecular weight = 176).

UV : λ max at 219 nm (for –C≡C–C≡C– unit) and 275 nm (for two conjugated system unit).

IR : bands at 3650 cm^{-1} –3100 cm^{-1} (–CH stretching); 2800cm^{1}– 3000 cm^{-1} (CH_3–CH_2–CH_2–C≡C–) ; 1600 cm^{-1} (strong) for carboxylate ion; 1445 cm^{-1} (medium) for –CH_2–C≡C– ; 1400 cm^{-1} (weak) for carboxylate ion; 1750 cm^{-1} for α–β–unsaturated ester; 2200 cm^{-1} (medium) for –C≡C– ;970 cm^{-1} for –CH=CH– and 1600 cm–1 for symmetrical diene.

IR spectrum of isolated lachrophyllum ester was found the finger print of IR of the pure sample of the ester.

PMR : A triplet at 1.0 (δ) for $-CH_3$; a hextet at 1.57 (δ) for $-CH_2-(CH_2-CH_2-CH_2-)$; at 6.13 (δ) for protons of $>C-C<^{COOR}_{H}$ and a singlet at 3.75 (δ) for $-OCH_3$.

Combustion analysis, UV, IR and PMR results correspond to the following structure :

$(CH_3-CH_3-CH_2-(C\equiv C-)_2\ CH=CH-COOCH_3$

The oil was also studied for antibacterial and antifungal properties by 'Filter paper disc diffusion technique' following Difco Laboratories Manual (6). The oil showed inactivity against *E. coli* and *Tricophyton mentagrophytes* . It showed slight activity against *S. Aureus*.

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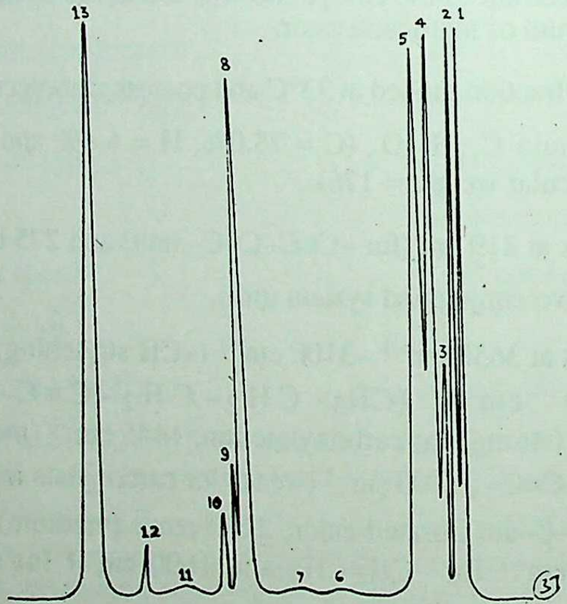


F 1 : GLC OF HYDRO CARBON FRACTION

TABLE - 1

GLC of Hydrocarbon fraction

Peak No.	Compound identified	Mode of identification
1.	α-pinene	IR, T_R
2.	β-pinene	IR, T_R
3.	myrcene	IR, T_R
4.	ocimene	IR, T_R
5.	limonene	IR, T_R
6.	u.i.	
7.	u.i.	
8.	caryophyllene	IR, T_R
9.	u.i.	
10.	humulene	IR, T_R
11.	u.i.	
12.	u.i.	
13.	cubebne	IR, T_R

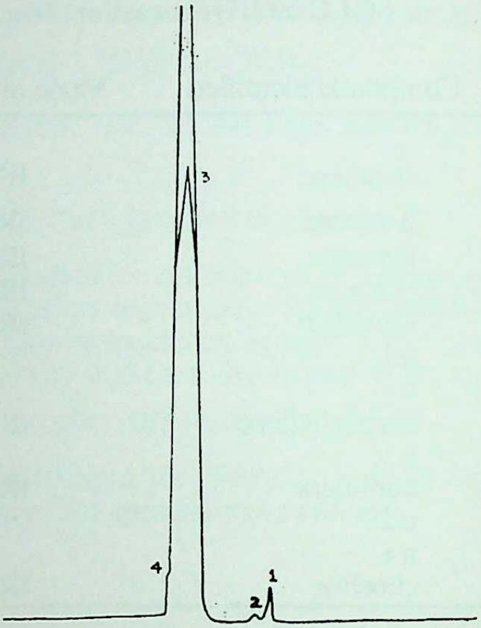
In this table u.i. denotes unidentified peaks.
IR denotes Infra red spectroscopy.
T_R denotes relative retention times.

TABLE - 2

GLC of Oxygenated fraction

Peak No.	Compound identified	Mode of identification
1.	u. i.	
2.	u. i.	
3.	lachrophyllum ester	IR, T_R
4.	u. i.	

In this table u.i. denotes unidentified peaks.
IR denotes Infra red spectroscopy.
T_R denotes relative retention times.



F 2 : GLC OF OXYGENATED FRACTION

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ON CONVEXITY IN CONVEX METRIC SPACES WITH APPLICATION

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1. ABSTRACT

In this paper we prove a generalization of a fixed point theorem from [7] for Takahashi convex metric spaces. For this class a characterization of convex hull is given too.

Key words and phrases: Convex metric space, fixed point. AMS (MOS) Subject Classifications (1980) : 47H10.

2. INTRODUCTION

In 1970 Takahashi [5] introduced the definition of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces. Subsequently, Machado [3], Talman [6], Naimpally and Singh [4], Guay and Singh [1], Hadžiac´ and Gajic´ [2], among others have obtained additional results in this setting.

This paper is a continuation of these investigations. Without additional conditions on convex structure we are able to give a characterization of convex hull. The major fact here is that the diameter is invariant under passage to so defined convex hull. In second part, for Takahashi convex metric spaces, we prove a generalization of a fixed point theorem from [7].

3. PRELIMINARIES

DEFINITION 1. Let X be a metric space and I be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$, $\lambda \in I$,

$$d(u, W(x,y,\lambda) \leq \lambda\, d(u,x) + (I-\lambda)\, d(u, y), \text{ for all } u \in X.$$

X together with a convex structure is called a convex metric space. Clearly any convex subset of a normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (I-\lambda) y$.

DEFINITION 2. Let X be a convex metric space. A nonempty subset K of X is said to be convex if and only if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Takahashi has shown ([15]) that open and closed balls are convex, and that the arbitrary intersection of convex sets is convex too.

For arbitrary $A \subset X$ let

$$(1) \qquad \widetilde{W}(A) := \{W(x,y,\lambda) : x,y \in A, \lambda \in I\}.$$

It's easy to see that $\widetilde{W} : P(X) \to P(X)$ is a mapping with properties:

i) $A \subset \widetilde{W}(A)$, for $A \subset X$;

ii) $A \subset B$ implies $\widetilde{W}(A) \subset \widetilde{W}(B)$, for $A, B \in P(X)$

iii) $\widetilde{W}\left(A \cap B\right) \subset \widetilde{W}(A) \cap \widetilde{W}(B)$, for any $A, B \in P(X)$.

Using this notation we can say that K is convex iff $\widetilde{W}(K) \subset K$.

A few additional definitions will be needed too.

DEFINITION 3. A convex metric space X will be said to have Property (C) iff every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

Remark. Every weakly compact convex subset of a Banach space has Property (C).

DEFINITION 4. Let X be a convex metric space and A be a nonempty, closed, convex, bounded set in X. For $x \in X$ we set

$$r_X(A) = \sup_{y \in A} d(x, y),$$

$$r(A) = \inf_{x \in A} r_X(A).$$

We then define $A_C = \{x \in A : r_X(A) = r(A)\}$ to be the centre of A. We denote the diameter of a subset A of X by

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

DEFINITION 5. A point $x \in A$ is a diametral point of A iff $\sup_{y \in A} d(x,y) = \delta(A)$.

DEFINITION 6. A convex metric space X is said to have normal structure iff for each closed bounded convex subset A of X, which contains at least two points, there exists $x \in A$ which is not a diametral point for A.

Remark. it is clear that any compact convex metric space has normal structure.

4. RESULTS

Now, let us recall:

DEFINITION 7. The convex hull of a set A $(A \subset X)$ is the intersection of all convex sets in X containing A and it is denoted by conv A.

It is obvious that if A is a convex set then $\widetilde{W}^n(A) = \widetilde{W}(\widetilde{W}(\ldots \widetilde{W}(A) \ldots) \subset A$ for any $n \in N$.

If we set :

$$A_n = \widetilde{W}^n(A), \qquad (A \subset X),$$

the sequence $\{A_n\}_{n \in N}$ is increasing so lim sup A_n exists and

$$\limsup A_n = \liminf A_n = \lim A_n = \bigcup_{n=1}^{\infty} A_n.$$

PROPOSITION 1. Let X be a convex metric space. Then

$$\text{(2)} \qquad \operatorname{conv} A = \lim A_n = \bigcup_{n=1}^{\infty} A_n, \qquad (A \subset X),$$

PROOF. If $x, y \in \bigcup_{n=1}^{\infty} A_n$, then there exists a positive integer n_0 (say) such that $x, y \in A_{n_0}$. So, for $\lambda \in I$, $W(x, y, \lambda) \in A_{n_0+1} \subset \bigcup_{n=1}^{\infty} A_n$. Thus $\bigcup_{n=1}^{\infty} A_n$ is convex and contains A. Further for any convex set C containing A, $\widetilde{W}^n(A) \subset C$ for every $n \in N$, i.e., $\bigcup_{n=1}^{\infty} A_n \subset C$. So, $\bigcup_{n=1}^{\infty} A_n = \operatorname{conv} A$.

In the remainder of the paper (X, d) will denote a convex metric space.

PROPOSITION 2. For any subset A of (X, d)

$\delta(\operatorname{conv} A) = \delta(A)$.

PROOF. Since $A \subset \operatorname{conv} A$ then $\delta(A) \leq \delta(\operatorname{conv} A)$. Now, let x and y be in conv A. If $x, y \in A$ it is obvious that $d(x,y) \leq \delta(A)$, so let one of them, for instance x, be in conv $A \setminus A$ and the second one from A.

Since $x \in$ conv A there exists $n_0 \in N$ such that $x \in \widetilde{W}^{n_0}(A)$. But it means that there exist $x_1, x_2 \in \widetilde{W}^{n_0-1}(A)$, $\lambda_1 \in [0, 1]$ so that $x = W(x_1, x_2, \lambda_1)$ and then :

$$d(x, y) = d(W(x_1, x_2, \lambda_1), y) \leq \lambda_1 d(x_1, y) + (1 - \lambda_1) d(x_2, y).$$

By induction one can see that there exist subset $\{\tilde{x}_1\}_{i\in I} \subset A$ (I finite set) and $\{\alpha_i\}_{i\in I}$, $\alpha_i \geq 0, \Sigma\alpha_i = 1,$ such that

$$d(x, y) \leq \sum_{i\in I} \alpha_i d(\tilde{x}_i, y).$$

Since

$$d(\tilde{x}_i, y) \leq \delta(A) \qquad \text{for } i\in I$$

we prove that :

$$d(x, y) \leq \delta(A). \tag{3}$$

Similarly one can prove that (3) is valid even in the case when $y \in$ conv A.

5. A FIXED POINT THEOREM

THEOREM. Let (X,d) be a metric space with continuous convex structure and let K be a closed convex bounded subset of (X, d) with normal structure and Property (C).

If $A: K \rightarrow K$ is a continuous mapping such that for $x,y \in K$

$$d(Ax,Ay) \leq max\{d(x,y), d(x, Ax), d(y,Ay)\ d(x, Ay), d(y, Ax)\} \tag{4}$$

then A has a fixed point.

PROOF, Let F be the family of all nonempty closed convex subset $F \subset K$ so that $A(F) \subset F$. Then F is nonempty since $K \in F$. Partilally order F by inclusion. Let $S=\{F_i\}_{i\in\Delta}$ be a decreasing chain in F.

From Property (C) it follows that $F_o = \bigcap_{i\in I} F_i \neq \phi$. So $F_0 \in F$.

Thus any chain in F has a greatest lower bound and by Zorn's Lemma there is a minimal member F in F. We claim that F is a singleton set. If not, then as shown by Takahashi [5] the center of F, denoted by F_c, is nonempt proper closed convex subset of F. It's easy to see that :

$$\delta(F_c) \leq r(F) \leq \delta(F).$$

Let us define a sequence :

$$F_0 = F_c \text{ and}$$

$$F_{k+1} = conv\,(F_k \cup A\,(F_k)),\ k = 0, 1, \ldots\ldots\ldots$$

Obviously $F_k \subset F_{k+1}$, (k = 0, 1, 2, ...) we shall prove by induction that :

$$(5) \qquad \delta_k = \delta(F_k) \leq r\,(F) = r \qquad \text{for any } k \in N.$$

For k=0 (5) is valid. Suppose that it is valid for k=0,1,2,...,....,m and let us prove that (5) is valid for $k = m+1$ too.

By definition of δ(F) for any sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$ ($n \in N$), $\lim\limits_{n \to \infty} \varepsilon_n = 0$ there exist $\tilde{x}_n, \tilde{y}_n \,\varepsilon F_{m+1}$ so that

$$\delta_{m+1} - \epsilon_n \leq d\,(\tilde{x}_n, \tilde{y}_n).$$

By Proposition 2 it is sufficient to analyse next three cases :

i) $\tilde{x}_n, \tilde{y}_n \,\varepsilon F_m$ $\qquad (n = 1, 2, \ldots)$,

ii) $\tilde{x}_n = x_n,\ \tilde{y}_n = Ay_n,\ (x_n, y_n, \varepsilon F_m,\ n = 1, 2, ..)$

iii) $\tilde{x}_n = Ax_n,\ \tilde{y}_n = Ay_n,\ (x_n, y_n, \varepsilon F_m,\ n = 1, 2, ..)$

At first case it is obvious that $\delta_{m+1} \leq$ r. So let us see the second one. For any $x \in F_0$ we have that:

$$(6) \qquad d\,(x, Ax) \leq r.$$

We shall suppose that (6) is valid for $x \,\varepsilon F_k$ $\{k = 0, 1, \ldots m-1\}$ and prove that it is valid for k=m.

For any $x \in F_m$, by Proposition 1,

$$x \in \widetilde{W}^{n_o}\left(F_{m-1} \cup A\left(F_{m-1}\right)\right)$$

for some $n_o \in N$.

Then

$$d(x, Ax) \le \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + \sum_{j \in I_2} \gamma_j d\left(Ax_j, Ax\right), \tag{7}$$

for $x_j \in F_{m-1}, j \in I = I_1 \, U \, I_2$, (I–finite set) $I_1 \cap I_2 = \phi$

$$\sum_{j \in I} \gamma_j = 1, \; \gamma_j \ge 0 \text{ for } j \in I.$$

As in [7] it is sufficient to look only for the case when $\sum_{j \in I_1} \gamma_j \ne 0$. Further we have that :

$$d(x, Ax) \le \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + \sum_{j \in I_2^{(1)}} \gamma_j d\left(x_j, x\right) + \sum_{j \in I_2^{(2)}} \gamma_j d\left(x, Ax\right)$$

$$+ \sum_{j \in I_2^{(3)}} \gamma_j d\left(x_j, Ax_j\right) + \sum_{j \in I_2^{(4)}} \gamma_j d\left(x, Ax_j\right) + \sum_{j \in I_2^{(5)}} \gamma_j d\left(x_j, Ax\right)$$

where we suppose

for $i \in I_2^{(1)}$ that $d(Ax_j, Ax) \le d(x_j, x)$;

for $i \in I_2^{(2)}$ that $d(Ax_j, Ax) \le d(x, Ax)$;

for $i \in I_2^{(3)}$ that $d(Ax_j, Ax) \le d(x_j, Ax_j)$;

for $i \in I_2^{(4)}$ that $d(Ax_j, Ax) \le d(x, Ax_j)$;

for $i \in 2(5)$ that $d(Ax_j, Ax) \le d(x_j, Ax)$;

and $I_2^{(j)} \cap I_2^{(k)} = \phi$ for $k \ne j$.

Now using the hypothesis one can see that

$$d(x, Ax) \le \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + r \sum_{j \in I_2^{(1)} \cup I_2^{(3)}} \gamma_j + \sum_{j \in I_2^{(2)}} \gamma_j d\left(x, Ax\right)$$

$$+ \sum_{j\in I_2^{(4)}} \gamma_j d(x, Ax_j) + \sum_{j\in I_2^{(5)}} \gamma_j d(x_j, Ax).$$

Since, by induction similarly we have that

$$d(x, Ax_j) \leq \sum_{k\in J_j^{(1)}} \beta_k d(\hat{x}_k, x_j) + \sum_{k\in J_j^{(2)}} \beta_k d(x_k, Ax_j)$$

$$+ \sum_{k\in J_j^{(3)}} \beta_k d(\hat{x}_k, A\hat{x}_k) + \sum_{k\in J_j^{(4)}} \beta_k d(x_j, A\hat{x}_k)$$

$$+ \sum_{k\in J_j^{(5)}} \beta_k d(\hat{x}_k, A\hat{x}_j)$$

for $\hat{x}_k \in F_{m-1}, k \left(\in J_j = \bigcup_{i=1}^{5} J_j^{(i)}, \sum_{k\in J_j} \beta_k = 1, \beta_k \geq 0, k\in J_j, \sum_{k\in J_j^{(1)}} \beta_k \neq 0 \right).$

So

$$d(x, Ax_j) \leq r \text{ and}$$

$$d(x, Ax) \leq \sum_{j\in I_1} \gamma_j d(x_j, Ax) + \left(\sum_{j\in I_2^{(1)}} + \sum_{j\in I_2^{(3)}} + \sum_{j\in I_2^{(4)}} \right) g$$

$$+ \sum_{j\in I_2^{(2)}} \gamma_j d(x, Ax) + \sum_{j\in I_2^{(5)}} \gamma_j d(x_j, Ax) \tag{8}$$

After not more than n_0- steps we shall have that :

$$d(x, Ax) \leq \sum_{j\in I^*} \gamma_j{}^* d(v_j, Ax) + \gamma_o{}^* r, \quad for\ \gamma_i{}^* \geq 0 \ \ i\in \{0\} \cup I^*$$

$\gamma_0{}^* + \sum_{j\in I^*} \gamma_j{}^* = 1$ *and* $v_j \in F_0, j\in I^*$. Since F_0 is centre we have that

$$d(v_j, Ax) \leq r$$

which implies that :

$$d(x, Ax) \leq r \qquad \text{for all } x\in F_m.$$

Similarly one can prove that $d(x, Ay) \leq r$ for all $x, y \in F_m$

so in the second case we have

$$\delta_{m+1} - \varepsilon_n \leq d(\tilde{x}_n, \tilde{y}_n) = d(x_n, Ay_n) \leq r, n \in N$$

and consequently

$$\delta_{m+1} \leq r.$$

Using (4) it is easy to prove this inequality in the case iii). So for any $m \in N$

$$\delta_m \leq r.$$

Let us define $F\infty := \bigcup_{k=0}^{\infty} F_k$

F_0 is nonemty set. So $F\infty$ is nonempty too.

Since $\delta(F\infty) < r < \delta(F)$, $F\infty$ is closed proper subset of F. Moreover, W is continuous and this guarantees that the closure of convex set is convex. Further, mapping A is continuous so:

$$A\left(F\infty\right) \subset F\infty$$

Therefore $F\infty \subset F$, a contradiction to the minimality of F. Hence F consists of a single element which is a fixed point for A.

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THERMAL STRESS IN A SEMI-INFINITE ROD OF NONHOMOGNEOUS MATERIAL

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ABSTRACT

Distribution of thermal stress in a semi-infinite elastic rod of nonhomogeneous material has been found. The density and the Young's modulus of the material are supposed to be functions of position. A constant temperature has been maintained at the end of the rod. Distribution of stress has been shown graphically for particular values of the parameters involved.

Key words : Thermal Stress, Young's modulus, Nonhomogeneous material, Laplace transform.

INTRODUCTION

The problem of determining thermal stresses in a elastic body has been gaining increasing importance because there appear many practical problems where mechanical structures are subjected to high temperature and pressure. Barrekette [1], Lal [3] and Nowinski [4] considered statical problems. But due to complexity involved in the governing differential equations it becomes quite difficult to solve thermoelastic problems when time appears as a variable. The problem of determining stress in rods is usually based on a one-dimensional stress-strain law and the hypothesis that plane sections of the rod remain plane after heating. The dynamical thermoelastic problem had been solved by Roychaudhuri [6] where he found the distribution of thermal stress in a thin rod of finite length one end of which was maintained at a constant temperature, the material of the rod being homogeneous. As regards nonhomogeneous problems, the governing differential equations become little more complicated and as a result sometimes simplifying assumptions are adopted in order to avoid such complications. The technique of Laplace transform is used, but due to natural difficulties in inversion, asymptotic expansion is followed to get solution of the problem valid for small time or limiting values of parameters are assumed. Latter type of non-homogeneous problem has been considered by Pal [5]. In the present investigation an attempt has been made to find thermoelastic stress distribution in a semi-infinite rod the end of which has been maintained at a constant temperature. The material of the rod has been assumed nonhomogeneous in the sense that both young's modulus E and the density ρ of the material are functions of position.

FORMULATION OF THE PROBLEM

Choosing x-axis along the axis of the rod let the position of the rod be given by $x \geq 0$. The differential equation of motion is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots (1)$$

where $\rho = \rho(x)$ is the density of the material and $u = u(x,t)$ is the longitudinal displacement of the centroid of the cross-section, which in the equilibrium state is at a distance x from the origin. The $\sigma = \sigma(x,t)$ is given by

$$\sigma = E\left(\frac{\partial u}{\partial x} - \alpha T\right) \qquad \ldots (2)$$

where $E = E(x)$ is the Young's modulus, $T = T(x,t)$ is the temperature at time t at α distance x from the end and α is the coefficient of linear expansion of the material of the rod.

Let the nonhomogeneity of the material be characterised by

$$E = E_o e^{-az} \qquad \dots (3)$$

$$\rho = \rho_o e^{-az}, \; a \neq o$$

where $z = x/L$ ... (4)

and L has the dimension of length.

In view of (3), substitution of (2) into (1) leads to

$$\frac{\partial^2 u}{\partial z^2} - a\frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial \tau^2} = \alpha L \left(\frac{\partial T}{\partial z} - aT\right) \qquad \dots (5)$$

$$\text{where } \tau = \frac{ct}{L}, \; c^2 = E_o/\rho_o. \qquad \dots (6)$$

The differential equation for the determination of temperature $T(x,t)$ at any point x at time t is

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}, \; x > 0, t > 0 \qquad \dots (7)$$

The quantity k generally depends upon the thermal conductivity, specific heat and the density of the material. For the sake of simplicity in the solution of (7) we assume that k is a constant.

Using (4) and (6), (7) becomes

$$\frac{\partial T}{\partial t} = h\frac{\partial^2 T}{\partial z^2} \qquad \dots (8)$$

Where $h = k/cL$.

The initial and boundary conditions of the problem are

$$u(z, 0) = 0 = \frac{\partial}{\partial \tau} u(z, 0), \; z > 0 \qquad \dots (9)$$

$$T(z, 0) = 0 \qquad \ldots (10)$$

$$\frac{\partial u}{\partial z} = \alpha L T, \; z = 0, \; \tau > 0 \qquad \ldots (11)$$

$$T(0, \tau) = T_o, \; \tau > 0 \qquad \ldots (12)$$

In addition to the above conditions we must have the regularity condition that the stress σ, displacement u and temperature T are bounded for all z and τ.

SOLUTION OF THE PROBLEM

Let us denote the Laplace transform with respect to τ, of any function $g(z,\tau)$ by $\overline{g}\,(z,p)$ where p is the transform parameter.

With this notation, Laplace transforms of (5) and (8), taking into account of initial conditions (9) and (10), yield

$$\frac{\partial^2 \overline{u}}{\partial z^2} - a\frac{\partial \overline{u}}{\partial z} - p^2\,\overline{u} = \alpha L\left(\frac{\partial \overline{T}}{\partial z} - a\overline{T}\right) \qquad \ldots (13)$$

and

$$\frac{\partial^2 \overline{T}}{dz^2} = \frac{p}{h}\,\overline{T} \qquad \ldots (14)$$

Laplace transform of boundary conditions (11) and (12) yield

$$\frac{\partial \overline{u}}{\partial z} = \alpha L\overline{T}, \; z = 0 \qquad \ldots (15)$$

$$\overline{T}\,(O,p) = \frac{T_0}{p} \qquad \ldots (16)$$

Solution of (14) with boundary condition (16) is given by Spiegel [7],

$$\overline{T}\,(z,p) = \frac{T_o}{p}\,e^{-\sqrt{\frac{p}{h}}\,z} \qquad \ldots (17)$$

In view of (17), differential equation (13) may now be solved for $\bar{u}$ as

$$\bar{u}\,(z,p\,) = A\,e^{\lambda z} - \frac{\alpha L T_o}{p}\,\frac{\left(\sqrt{\frac{p}{h}}+a\right)e^{-\sqrt{\frac{p}{h}}\,z}}{a\sqrt{\frac{p}{h}}+\frac{p}{h}-p^2} \qquad \ldots (18)$$

where the regularity condition $\bar{u}\ (\infty\ ,p) = 0$ has been used. The quantity λ is given by

$$\lambda = \mu - \sqrt{p^2+\mu^2} \qquad \ldots (19)$$

where $\mu = a\,/\,2$.

The constant A may be determined by using (15) and (16). Finally, we obtain $\bar{\sigma}(z,p)$ as

$$\bar{\sigma}(z,p) = \alpha\,E_o E\,e^{-az}\left[e^{-z\sqrt{p/h}} - e^{\lambda z}\right]\frac{p}{\frac{p}{h}+a\sqrt{\frac{p}{h}}-p^2} \qquad \ldots (20)$$

If β_1, β_2, β_3, are the roots of the cubic

$$y^3 - b^2 y - a b^2 = 0 \qquad \ldots (21)$$

where $b = 1/h$ and $y = \sqrt{p/h}$, then (20) may be written as

$$\bar{\sigma}\,(z,p\,) = \alpha\,E_o T_o\,e^{-az}\left[e^{\lambda z}\sum_{i=1}^{3}\frac{L_i}{\sqrt{p}\left(\sqrt{p}+\beta_i\,\sqrt{h}\right)}\right.$$

$$\left. - \frac{e^{-z\sqrt{p/h}}}{h}\sum_{i=1}^{3}\frac{M_i}{p\left(\sqrt{p}-\beta_i\,\sqrt{h}\right)}\,\frac{1}{p}\,e^{-z\sqrt{p/h}}\right] \qquad \ldots (22)$$

where

$$L_i = \frac{\beta_i^{\,2}}{\left(\beta_j-\beta_i\right)\left(\beta_k-\beta_i\right)}$$

$$M_i = \frac{a-\beta_i}{(\beta_j-\beta_i)(\beta_k-\beta_i)},\ i \neq j \neq k,\ i, j, k = 1, 2, 3$$

Equation (22) may now be used to find the stress σ (z, τ) at any point z at time τ by applying inverse Laplace transform [cf. Erdelyi [2]]

$$\sigma(z,\tau) = \alpha E_o T_o e^{-az/2}\left[G(\tau-z)H(\tau-z) + \int_z^\tau f(u,z)\,G(\tau-u)\,du\right]$$

$$+\alpha E_o T_o\, e^{-az}\left[g(\tau,z) - \text{erfc}\left(z/2\sqrt{h\tau}\right)\right] \qquad \dots (23)$$

where

$$f(\tau,z) = -\frac{\mu z}{\sqrt{\tau^2-z^2}} J_1\left(\mu\sqrt{\tau^2-z^2}\right)H(\tau-z)$$

$$G(\tau) = \sum_{i=1}^{3} L_i\, e^{\beta_i^2 h\tau}\,\text{erfc}\left(\beta_i\sqrt{h\tau}\right)$$

$$g(\tau,z) = \sum_{i=1}^{3}\frac{M_i}{h^{3/2}\beta_i}\left[\text{erfc}\left(z/2\sqrt{h\tau}\right) - e^{\beta_i z + \beta_i^2 h\tau}\,\text{erfc}\left(\frac{z}{2\sqrt{h\tau}} + \beta_i\sqrt{h\tau}\right)\right]$$

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt$$

and H (τ) is Heaviside's step function.

NUMERICAL RESULTS

To get an idea about the effect of nonhomogeneity as stipulated in (3) on the stress, variations of stress with position and time as given by (23) has been numerically computed. Although the roots $\beta_1, \beta_2, \beta_3$ of the cubic in (21) may be found by any standard method, the restriction $b = 3\sqrt{3a}$ makes the procedure quite simple. In our numerical evaluation we have used this condition and get $\beta_1 = -5.63816\, a$, $\beta_2 = 4.59625\, a$, $\beta_3 = 1.04190\, a$.

In figures 1 and 2 we have shown the variations of

$$p = \frac{\sigma}{\alpha E_o T_o}$$

With τ over the range 0 to 6.0 for τ. In Fig. 1 the parameter a in (3) has been fixed at 0.2 and the variations of stress with positions have been shown. It is observed from the graph that at a particular position stresses diminish with time while within the range of τ considered the magnitude of stress at a particular time for points on the right of the rod is greater than that for the points in the left. From the nature of the graph it is apparent that for sufficiently large values of z he stress becomes negligible, which is in conformity with the regularity condition. In Fig. 2 we have plotted the variations of P with time keeping z fixed at 2. It is observed that with the increase of a the stress diminishes, while the behaviour of the stress remaining the same.

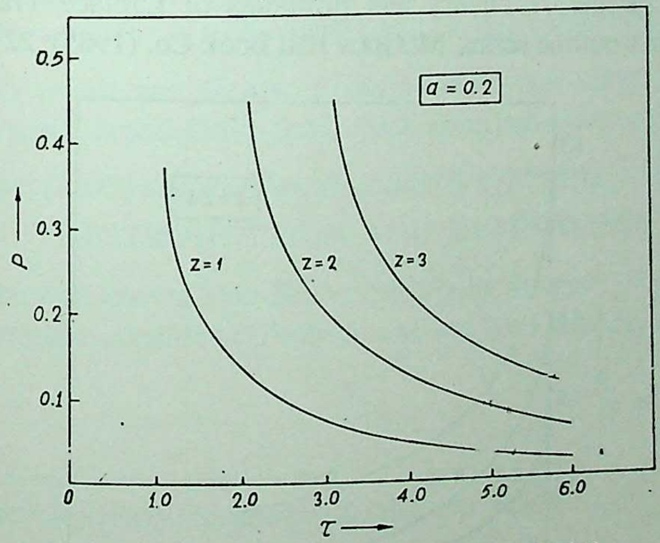

Fig.1 Variation of stress with time (for fixed a)

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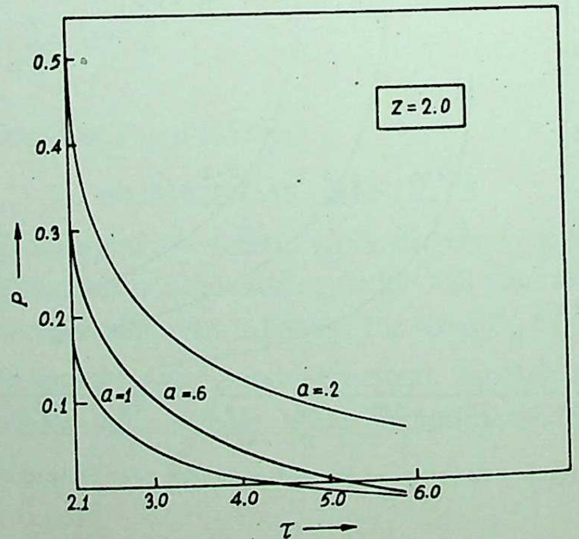

Fig. 2 Variation of stress with time (for fixed Z)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (1989), 57-67

COINCIDENCES AND FIXED POINTS OF MEIR-KEELER TYPE CONTRACTIVE MAPPINGS ON MENGER SPACES

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ABSTRACT

In this paper the notion of Meir-Keeler type contractive triplet of mappings on a Menger space is introduced and some coincidence and fixed point theorems for such mappings are established. The present results generalize a number of well-known results in fixed point theory.

Key words and phrases : Meir-Keeler type contractive triplet, Menger space, probabilistic metric space, fixed point, coincidence point.

AMS (MOS) Subject Classifications (1980) : 54H25, 54E99

INTRODUCTION AND DEFINITIONS

The well-known Meir-Keeler fixed point theorem [8] proved under the contractive condition (1) with $A = M$ and $T = I$ (Identity mapping) has

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been applied, extended and studied, among others by Chung [2] Ćiric´ [3] Maiti-pal [6], Matkowski-Wegrzyk [7], Park [9] Park-Bae [10], Park-Moon [11], Park-Rhoades [12], Rao-Rao [13], Rhoades [14], Shisheng [16], Singh [17], Singh-Virendra [20] and Yen-Chung [21]. For fixed point theorems for three mappings on a metric space satisfying Meir-Keeler type contractive conditions, refer to Park-Moon [11] and Rhoades [14]. However, fixed point theorems of these authors and several others, except of course [13] and [17], require continuity condition on the mapping(s) under consideration.

Recent coincidence theorems (on metric spaces) of Park [9] and Singh [17] extend and unify a coincidence theorem of Goebel [4], Meir-Keeler fixed point theorem [8] and a few fixed point theorems for contractive mappings.

In this paper we introduce Meir-Keeler type contractive conditions in Menger spaces and establish coincidence and fixed point theorems for three mappings, wherein either we use no continuity conditions or a very mild continuity condition is used.

In all that follows, A stands for an arbitrary nonempty set, (M,d) for a metric space, X for a Menger space (X,f,t) where t is continuous and satisfies $t(x,x) \geq x, x \in [0,1]$. By $C(P,T)$ we denote the set of coincidence points of mappings P and T, i.e., $C(P,T) = \{z \in A : Pz = Tz\}$.

Definition 1. A probabilistic metric space (PM-space) is an ordered pair (X,f) consisting of a nonempty set X and a mapping, f form $X \times X$ to L the collection of distribution functions. We shall represent the value of f at $(u,v) \in X \times X$ by $F(u,v)$. The functions $F(u,v)$ are assumed to satisfy the following conditions :

(a) $F(u,v;x) = 1$ for all $x > 0$, iff $u = v$;

(b) $F(u,v;0) = 0$ for all $u,v \in X$;

(c) $F(u,v) = F(v,u)$ for all $u, v \in x$;

(d) if $F(u, v; x) = 1$ and $F(v, w; y) = 1$ then

$F(u, w; x+y) = 1.$

A Menger space is a triplet (X, f, t) where (X, f) is a PM-space and t-norm (or T-norm [15]) t is such that (d) is replaced by

(d′) $F(u,w; x+y) \geq t\,(F(u,v; x), F\,(v,w,y))$

for all $x, y \geq 0$. For the details of topological preliminaries, refer to schweizer and Sklar [15].

Let P,Q,T be mappings on A with values in M or X as per situation. Consider the following conditions :

(1) For a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ Such that for $u, v \in A$, $\varepsilon \leq d\,(Tu, Tv) < \varepsilon + \delta +$ implies $d\,(Pu, Pv) < \varepsilon$.

(2) For a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for u, $v \in A$,

$\varepsilon \leq \max\{d\,(Tu, Tv), d\,(Tu, Pu), d\,(Tv, Qv), 1/2\,[d(Tu, Qv) + d\,(Tv, Pu)]\}$

$< \varepsilon + \delta$ implies $d(Pu, Qv) < \varepsilon$.

The condition (1) has been used to generalize Goebel's coincidence theorm [4] by Park [9], while Park-Bae [10], using the condition (1) with A = M and PT = TP, have unified the fixed point theorems of Jungck [5] and Meir-Keeler [8]. A recent fixed point theorem of Park-Moon [11, Th.3.1], proved under the condition (2) with A = M, T continuous and commuting with each of P and Q extends and unifies a number of Meir-Keeler type fixed point theorems. Park-Moon [11] have also given a nice comparison of Meir-Keeler type contractive mappings. Note that (2) includes (1). Now, akin to (2) we introduce a Meir-Keeler type contractive condition for three mappings in a Menger space. Our results proved under this condition extend and include fully a number of results, among others, from [3-6], [8-10], [18], [19], and partially from [11-13], [17] and [20].

Let $\quad \min\{F(Tu, Tv; x), F(Tu, Pu; x), F\,(Tv, Qv; x)$

$F(Tu, Qv; 2x), F(Tv, Pu; 2x)\} = K(u, v; x)$ *for* u,v *in* A *and* $x > 0$.

Definition 2. Three selfmappings P,Q,T on X will be called a Meir-Keeler type contractive triplet if given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

(3) $0 = K(u, v; \varepsilon) < K(u,v; \varepsilon + \delta) = 1$ implies

$F(Pu, Qv; \varepsilon) = 1$.

Definition 3. Let P,Q,T be selfmappings on X If there exists a point u_0 in X and a sequence $\{u_n\}$ in X such that

$$Tu_{2n+1} = Pu_{2n}, Tu_{2n+2} = Qu_{2n+1}, n = 0,1,2,.....$$

then the space X is called $(P, Q; T(u_0))$ - orbitally complete with respect to μ_0 or simply $(P,Q; T)$ - orbitally complete if the closue of $\{T_{u_n} : n = 1,2,.......\}$ is complete.

Definition 4. The mapping T is called $(P,Q,; T(u_0))$ - orbitally continuous if the restriction of T on the closure of $\{Tu_n : n = 1,2,......\}$ is continuous.

Definition 5. Two selfmapings P and T on X will be called weakly commuting if,

$F(PTu, TPu; x) \geq F(Tu, Pu; x)$ *for all* $u \in X$ *and* $x > 0$.

RESULTS

The following lemma is motivated by $S.-S.$ Chang [1].

Lemma. Let $\{y_n\}$ be a sequence in X such that $\lim_n F(y_n, y_{n+1}; \varepsilon) = 1$ for any $\varepsilon > 0$. If $\{Y_n\}$ is not Cauchy then there exist $\varepsilon_0 > 0$ and two sequences of positive integers, $\{m_i\}$ and $\{n_i\}$, such that

(a) $m_i > n_i + 1, i \to \infty, n_i \to \infty;$

(b) $F(y_{m_i}, y_{n_i}; \varepsilon_0) \leq 1 - \lambda, \quad 1 = 1, 2, ...,$

(c) $F(y_{m_i-1}, y_{n_i}; \varepsilon_0) > 1-\lambda, \quad i = 1, 2, \ldots$

Proof. Since $\lim_n F(y_n, y_{n+1}; \varepsilon) = 1$, for any $\varepsilon_0, \lambda > 0$, there exists a positive integer N such that $F(y_n, y_{n+1}; \varepsilon_0) > 1-\lambda$ for $n \geq N$. As the sequence is not cauchy, we have for $\varepsilon_0, \lambda > 0$,

$$F(y_{n_1}, y_{p_1}; \varepsilon_0) \leq 1-\lambda, p_1 > n_1 > N_1 > N.$$

Let $S = \{p \in I^+ : F(Y_{n_1}, Y_p; \varepsilon_0) \leq 1-\lambda,\ p > n_1\}$, I^+ being the set of positive integers. Then S is clearly nonempty. Let $p_0 \in S$ such that $p_0 \leq n_1 + 1$. Then $n_1 + 1 \geq p_0 > n_1 > N$, *i.e.*, $p_0 = n_1 + 1$. Thus,

$F(y_{n_1}, y_{p_0}; \varepsilon_0) = F(y_{n_1}, y_{n_1+1}; \varepsilon_0) > 1-\lambda$, a contradiction, implying $p > n_1+1, p \in S$. Now we can find a minimal integer $m_1 \in A$ such that $m_1 < n_1 + 1$, $F(y_{n_i}, y_{m_i}; \varepsilon_0) \leq 1 - \lambda$ and $F(y_{n_i}, y_{m_i}; \varepsilon_0) < 1-\lambda$. Repeated use of these arguments gives the desired conclusions.

Theorem 1. Let P, Q, T be a Meir-Keeler type contractive triplet on A with values in X. If there exists a point u_0 in A such that X is $(P, Q; T(u_0))$ – orbitally complete then P, Q, T have a coincidence, i.e., there exists a point z in A such that $Pz = Qz = Tz$. Further, if $A = X$ and for all $u, v \in X, x > 0$, either

(i) $F(Qu, Tv; x) \geq \min\{F(v, Qu; x), F(v, Tu: x)\}$

or

(ii) $F(Pu, Tv; x) \geq \min\{F(v, Tu; x), F(v, Pu; x)\}$ then

P, Q, T have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

Proof. Condition (3) is equivalent to

(4) $F(Pu, Qv; x) > \min\{F(Tu, Tv; x), F(Tu, Pu; x), F(Tv, Qv; x), F(Tu, Qv; 2x), F(Tv, Pu; 2x)\}$.

Two cases arise.

Case I. suppose $Tu_n = Tu_{n+1}$ and $Tu_{n+1} \neq Tu_{n+2}$ for some odd integer *n*. By (4)

$$F(Tu_{n+2}, Tu_{n+1}; x)$$

$$= F(Pu_{n+1}, Qu_n; x)$$

$$> \min\{F(Tu_{n+1}, Tu_n; x), F(Tu_{n+1}, Tu_{n+2}; x), F(Tu_n, Tu_{n+1}; x), F(Tu_{n+1}, Tu_{n+1}; 2x), F(Tu_n, Tu_{n+2}; 2x)\}$$

$= F(Tu_{n+1}, Tu_{n+2}; x)$, since by *(d')*

$$F(Tu_n, Tu_{n+2}; 2x) \geq \min\{F(Tu_n, Tu_{n+1}; x), F(Tu_{n+1}, Tu_{n+2}; x)\}.$$

But this is a contradiction.

Hence $Tu_{n+1} = Tu_{n+2}$, *i.e.*, in general

$$Tu_{n+i} = Tu_n, i = 1, 2, \ldots$$

Taking $u_n = z$, we have

(5) $Tu_n = Tu_{n+1} = Qu_n$, *i.e.*, $Tz = Qz$.

Let $Pz \neq Qz$, then from (4),

$F(Pz, Qz; x) > F(Pz, Qz; x)$, a contradiction.

Thus, $Pz = Qz$. Hence, from (5),

$$Pz = Qz = Tz.$$

The case when *n* is even may be dealt with similarily.

Case II. Suppose $Tu_n \neq Tu_{n+1}$ for all integers *n*. The application of (4) gives that $\{F(Tu_n, Tu_{n+1}; x)$ is an increasing sequence converging to a number $0 \leq r \leq 1$. Let $r < 1$.

From this we infer that for every $s \in (0, r)$ there exists a positive integer N such that $r - s \le F(Tu_n, Tu_{n+1}; x) < r$ for all $n \ge N$, since the convergence is monotone from the left.

Let $2k \ge N$, then by (4)

$$F(Tu_{2k+1}, Tu_{2k+2}; x) > \min\{F(Tu_{2k}, Tu_{2k+1}; x), F(Tu_{2k}, Pu_{2k}; x),$$
$$F(Tu_{2k+1}, Qu_{2k+1}; x), F(Tu_{2k}, Qu_{2k+1}; 2x),$$
$$F(Tu_{2k+1}, Pu_{2k}; 2x)\} = F(Tu_{2k}, Tu_{2k+1}; x).$$

Thus $F(Tu_{2k+1}, Tu_{2k}+2; x) > r$, a contradiction implying $r = 1$.

To prove that the sequence $\{Tu_n\}$ is a Cauchy sequence, suppose the contrary to be true. Then, by the Lemma, there exists $\varepsilon_0 > 0$ and two sequences of positive integers,

$\{m_i\}$ and $\{n_i\}$, such that

(a) $m_i > n_i + 1, n_i \to \infty, i \to \infty;$

(b) $F(Tu_{m_i}, Tu_{n_i}; \varepsilon_0/2) \le 1 - \lambda, i = 1, 2, \dots;$

(c) $F(Tu_{m_i-1}, Tu_{n_i}; \varepsilon_0/2) > 1 - \lambda, i = 1, 2, \dots.$

Thus using (b), (c) and passing on to the limits we have

$$F(Tu_{m_i}, Tu_{n_i}\, e_0) \ge \min\{F(Tu_{m_i}\, Tu_{m_i}-1; \varepsilon_0/2),$$
$$F(Tu_{m_i-1}, Tu_{n_i}; \varepsilon_0/2)\} > \min\{1, 1-\lambda\} = 1 - \lambda,$$

a contradiction.

Therefore $\{Tu_n\}$ is a Cauchy sequence. Since × is $(P, Q, T(u_0))$ – orbitally complete, there exists a z in × such that $Tu_n \to z$.

Now if (i) holds, then

$F(Qu_{2n+1}, Tz; x) \ge \min\{F(z, Qu_{2n+1}; x), F(z, Tu_{2n+1}; x)\};$ implying $Tz = z$.

And if (ii) holds then also $Tz = z$.

Also, since $Tu_m \neq Tu_{m+1}$, we get by (4),

$F(z, Qz; x) > F(z, Qz; x)$, giving $z = Qz$ which in trun, again with the application of (4), gives $z = Pz$.

Thus $\quad Pz = Qz = Tz = z$.

With suitable choice of P, Q, T in (3) we get probabilistic analogues of various well-known coincidence and fixed point theorems in metric spaces, some of them being due to Goebel [4], Meir-Keeler [8], Park [9], Park-Bae [10], Rao-Rao [13], Singh [17] and Singh-Virendra [20].

Theorem 2. Let $P, Q, T : X \to X$ be a Meir - Keeler type contractive triplet. If there exists a point u_0 in X such that X is $(P, Q; T(u_0))$ – orbitally complete and,

either (i) $PTz = TPz, z \in C(P, T)$;

or (ii) $QTz = TQz, z \in C(Q, T)$, then P,Q,T, have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

Proof. The uniquene of the fixed point of P, Q, T if it exists, is ensured by (4) with $A = X$. So, in view of Case I of the proof of Theorem 1, it is sufficient to show that P, Q, T have a common fixed point $Tz(=Pz=Qz)$. Without loss of generality we may assume that P and T commute at $z \in C(P, T)$.

Suppose (i) holds. Then

(6) $\quad PTz = TPz, = TTz, \text{ as } Tz = Pz.$

Now taking $TTz \neq Tz$, we have by (4),

$F(TTz, Tz; x) = F(PTz, Qz; x) > F(TTz, Tz; x)$, a contradiction.

Thus, in view of (6)

$TTz = Tz = PTz.$

Clearly $QTz = Tz$, since otherwise it leads to a contradiction. Hence $Tz(=Pz=Qz)$ is a common fixed point of P, Q, T. Hence $Tz\ (=Pz=Qz)$ is a common fixed point of P, Q, T.

Remark: It is interesting to note that every pair of mappings (P, T) commuting weakly on $C(P,T)$ is commuting on $C(P,T)$. Thus weak commutativity of P and T at a coincidence point z of P and T is equivalent to the commutativity of P and T at z. Therefore Theorem 2 holds even if commutativity of P and T (resp. Q and T) is replaced by weak commutativity of P and T (resp. Q and T) Thus the proof of the following theorem is obvious.

Theorem 3 . Let $P, Q, T, : X \to X$ be a Meir-Keeler type contractive triplet. If there exists a point u_0 in X such that X is $(P, Q,; T (u_0))$ – orbitally complete and either

(i) P and T are weakly commuting and $(P,Q;T(u_0)$ –orbitally continuous;

or

(ii) Q and T are weakly commuting and $(P, Q; T(u_0))$ –orbitally continuous, then P, Q, T have a unique common fixed point p (say) in X and for some $u_0 \in X$, the sequence $\{Tu_n\}$ converges to a point z in X such that $Tz = p$.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (1989), 68-77

LOGISTIC GROWTH UNDER RANDOM DRIVING FORCES

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ABSTRACT

The stochastic behaviour of non-equilibrium fluctuation in the logistic model of growth under random driving forces has been studied in the light of the 'fluctuation-dissipation' theorem of statistical mechanics.

Key words : Logistic growth equation, Carrying capacity, Non-equilibrium fluctuation, White-noise, Fluctuation-dissipation theorem.

AMS (MOS) subject classification (1980) : 92A17

1. INTRODUCTION

Logistic growth equation plays an unique place as the first non-linear equation for population growth. It finds wide application in different branches of biology. It is also the basic equation of non-linear autocatalytic reactions which is fundamental in understanding many phenomena of life processes [7]. A lot of works was done on this equation covering various aspect of population, but there are others to be studied from a different consideration.

In this paper we shall study the stochastic behaviour of non-equilibrium fluctuation in the logistic growth of a biological system under the influence of random driving force. The randomness appeared here is assumed to be internal, that is, due to the random interactions among the individuals of a single species [3]. Two cases namely, the state with constant carrying capacity and the other with time-varying capacity have been considered.

2. VERHULST GROWTH UNDER RANDOM DRIVING FORCE

A. *Constant Carrying Capacity*

The Verhulst (or Logistic) equation of growth of a single species population is given by

$$\frac{dN}{dt} = \alpha N - \beta N^2 = \alpha N \left(1 - \frac{N}{K}\right) \qquad \ldots (1)$$

where $N(t)$ is the population size at any time t, α is the intrinsic rate constant and $K = \alpha/\beta$ is the carrying capacity of the environment. Let us first assume that K or β is constant. This type of equation occurs in different cases of biophysical system [7]. The stochastic extension of the equation (1) is given by

$$\frac{dN}{dt} = \alpha N - \beta N^2 + \varepsilon(t) \qquad \ldots (2)$$

where $\varepsilon(t)$ is the random perturbation term which is due to the overall effect of the numerous microscopic, unknown or partially known internal fluctuation of the state variable N. We have called the randomness of perturbation to be internal in the sense that it is due to the internal effect of mutual interactions of the individuals of the same species without any consideration of the randomness of the environmental parameters. The random perturbation term $\varepsilon(t)$ is assumed to be a white noise satisfying the condition .

$$\langle \varepsilon(t) \rangle = 0, \ \langle \varepsilon(t_1)\, \varepsilon(t_2) \rangle = 2D_0\, \delta(t_1 - t_2) \qquad \ldots (3)$$

where < > represents the average over the ensemble of the stochastic process. D_0 is the diffusion coefficient. The equation (2) is a non-linear stochastic differential equation of non-linear Langevin type and there are some difficulties in solving such problem [6]. We shall, however, resort to the linearization method to study the characteristic stochastic behaviour of the system. The equation (2) is linearized as follows :

The equation for the average $\langle N \rangle$ is given by

$$\frac{d\langle N \rangle}{dt} = \alpha \langle N \rangle - \gamma \langle N^2 \rangle \qquad \text{...(4)}$$

or, approximately by

$$\frac{d\langle N \rangle}{dt} = \alpha \langle N \rangle - \gamma \langle N \rangle^2 \qquad \text{... (5)}$$

Where we have used the relation $\langle N^2 \rangle = \langle N \rangle^2$ as a first approximation.

Let us put $N(t) = \langle N \rangle + N_0(t)$ ... (6)

in equation (2) and subtracting (4) from it, we get the linear equation

$$\frac{dN_0}{dt} = \left(\alpha - 2\beta \langle N \rangle\right) N_0 + \varepsilon(t) \qquad \text{... (7)}$$

Where we have neglected second order terms of $N_0(t)$.

If $\sigma^2(t) = \left\langle (N - \langle N \rangle)^2 \right\rangle = \left\langle N_0^2 \right\rangle$

the $\sigma^2(t)$ satisfies the ordinary differential equation [2]

$$\frac{d}{dt}\sigma^2(t) = 2\left(\alpha - \beta \langle N \rangle\right)\sigma^2(t) + 2D_0 \qquad \text{... (8)}$$

For α > 0 and for large t, $\langle N \rangle \to 0$ so that the equation (8) takes the form

$$\frac{d\left\langle N_0^2(t)\right\rangle}{dt} = 2\alpha\left\langle N_0^2(t)\right\rangle + 2D_0 \qquad \text{... (9)}$$

Then $\left\langle N_0^2(t)\right\rangle\Big|_{t\,\to\,\text{lage}} = \dfrac{D_0}{\alpha}$ ... (10)

B. Quasi-steady State : Fluctuation

Let us now consider the situation of the quasi-steady state which corresponds to the state very near to the steady non-equilibrium state. Such a state is characterized by the slow variation of the drift coefficient $A(t)\left(\alpha - \beta\ \langle N\rangle\right)$ and the diffusion coefficient D_0 which is assumed here to be constant. In addition, the addmittance functionn $Z(i\omega,t)$ relating $N_0(t)$ and $e(t)$ of the stochastic equation (7) must be slowly varying function of time. Under these assumptions, the mean square fluctuation of the population size $N(t)$ is given by [1]

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\int_{-\infty}^{\infty}\left|\frac{A(t)}{\left(A(t)+i\omega\right)} + \frac{A(t)}{\left(A(t)+i\omega\right)^3}\right|^2 d\omega \qquad \text{... (11)}$$

The above formula is a generalization of the famous Nyguist's relation for quasi-stationary state [1]. For the slowly varying condition of the drift coefficient $A(t) = \left(\alpha - \beta\ \langle N\rangle\right)$, We must have

$$\left|\frac{\dot{A}(t)}{A(t)}\right| << 1 \qquad \text{... (12)}$$

or, $\left|\beta\ \langle N\rangle\right| << 1$ ... (13)

Under this condition let us now find the value of $\left\langle N_0^2(t)\right\rangle$. Now (11) can be reduced to the form

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\left(A^2(t)+\omega^2\right)} - \frac{2\dot{A}\cdot(t)}{\left(A^2(t)+\omega^2\right)^2}\right.$$

$$\left. + \frac{4\dot{A}(t)A^2(t)+\dot{A}^2(t)}{\left(A^2(t)+\omega^2\right)^3}\right]d\omega \qquad \dots (14)$$

Since $A(t) = \alpha - \beta\left\langle N\right\rangle$, $\dot{A}(t) = -\beta\left\langle N\right\rangle A(t)$

So

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\left[\int_{-\infty}^{\infty}\frac{d\omega}{A^2(t)+\omega^2} + 2\beta A(t)\langle N\rangle\int_{-\infty}^{\infty}\frac{d\omega}{\left(A^2(t)+\omega^2\right)^2}\right.$$

$$\left. + \left\{\beta^2\langle N\rangle^2 A^2 - 4\langle N\rangle A^3\beta\right\}\int_{-\infty}^{\infty}\frac{d\omega}{\left(A^2(t)+\omega^2\right)^3}\right] \qquad \dots (15)$$

Evaluating the integrals in (15), we have

$$\left\langle N_0^2(t)\right\rangle = D_0\left[\frac{1}{A(t)} - \frac{\beta\langle N\rangle}{2A^2(t)} + \frac{3\beta^2\langle N\rangle^2}{8A^3(t)}\right]$$

$$\simeq \frac{D_0}{\alpha}\left\langle 1 + \frac{\beta\langle N\rangle}{\alpha}\right\rangle$$

$$= \frac{D_0}{a} + \frac{D_0}{a^2}\beta\langle N\rangle \qquad \dots (16)$$

neglecting higher powers of $\beta \langle N \rangle$. Then for small $\beta \langle N \rangle$ under the condition (13), the result (16) for the quasisteady state agrees with the result (10) for large time *t*, as it should be.

3. VERHULST GROWTH UNDER RANDOM AND DETERMINISTIC DRIVING FORCES

A. *Periodic Carrying Capacity : Linearization*

Let us now consider the Verhulst equation (1) where the capacity *K* is now assumed to be time-dependent and this due to the variability of the environmental condition. Let us take *K*(*t*) to be of sinusoidal form

$$K(t) = K_0 (1 + b \cos \omega t) \qquad \text{... (17)}$$

Then the equation (1) becomes

$$\frac{dN}{dt} = \alpha N \left[1 - \frac{N}{K_0 \left(1 + b \cos \omega t \right)} \right] \qquad \text{... (18)}$$

The equation (18) can be scaled as follows

We write $N' = \frac{N}{N_0}$, $t' = \alpha t \cdot$ ... (19)

Then the equation (18) reduces to the form

$$\frac{dN'}{d t'} = N' \left[\frac{N}{\left(1 + b \cos \omega' t' \right)} \right] \qquad \text{... (20)}$$

where $\omega' = \frac{\omega}{\alpha}$ ... (21)

The steady state value for N' as seen from (20) is given by

$$N' + 1 = b \cos \omega' t' \qquad \text{... (22)}$$

For $b \ll 1$ and for long time interval the steady value of *N'* is one. Then writing $N' = 1+y$ and expanding about the steady value, the linear equation that results from (20) is given by [5]

$$\frac{dy}{dt'} + y = b\cos\,\omega' t' \qquad \text{... (23)}$$

The equation (23) is the basic deterministic equation and is a linear equation in y with a driving time-dependent force $b\cos\omega' t'$.

B. Stochastic Differential Equattion : Fluctuation

The stochastic extension of (23) is obtained by addition of random perrturbation term $\varepsilon(t')$ to the right hand side of (23).

$$\frac{dt}{dt'} + y = b\,\omega' t' + \varepsilon(t') \qquad \text{... (24)}$$

Where random noise $\varepsilon(t')$ is assumed to be a gaussian white noise satisfying the conditions

$$\left\langle \varepsilon(t_1')\right\rangle = 0,\ \left\langle \varepsilon(t_1')\,\varepsilon(t_1')\right\rangle = 2D\,\delta\left(t_1' - t_2'\right) \qquad \text{... (25)}$$

Where < > represents the average over the ensemble of the stochastic process. The equation (24) is an extension of Langevin equation driven by a periodic force $b\cos\omega' t'$ and a random force $\varepsilon(t')$. The stochastic behaviour of the system satisfying the equation (24) can be studied from the extended form of the famous 'fluctuation- dissipation theorem' of non-equilibrium statistical mechanics [4].

To find the correlation function for the system satisfying the stochastic equation (24) let us assume that the system has evolved from the state $x = 0$ at some very distance time past $t = -\infty$ under the action of the time-varying external force $b\cos\omega' t'$. Then following Lavenda [4] the auto - correlation function is given by

$$C\left(t', t_0'\right) = \left\langle y\left(t'\right) y\left(t_0'\right)\right\rangle$$

$$= D\exp\left\{-\left(t' - t_0'\right)\right\} = \int_{-\infty}^{t'} \exp\left\{-\left(t' - u\right)\right\} b\cos\,\omega' u\, du$$

$$\int_{-\infty}^{t'_0} exp\left\{-\left(t'_0-u\right)\right\} b\cos\ \omega' u\ du \qquad \text{... (26)}$$

$$= D\,exp\ \left(t'_0-t'\right)+b\,\frac{\left(\omega'\sin\ \omega' t' + \cos\ \omega' t'\right)}{1=\omega'}$$

$$b\,\frac{\left(\omega'\sin\ \omega' t'_0 + \cos\ \omega' t'_0\right)}{1+\omega'} \qquad \text{... (27)}$$

From (27) the average fluctuation $\left\langle y^{2}\left(t'\right)\right\rangle$ is given by

$$\left\langle y^{2}\left(t'\right)\right\rangle = D + \frac{b^2}{\left(1+\omega'\right)^2}\left(\omega'\sin\omega' t' + \cos\ \omega' t'\right)^2 \qquad \text{... (28)}$$

which consists of two parts. The first part is due to the random force and the second part is due to the driving force, i.e., due to the environmental variability coming through the time-dependent carrying capacity. The average of $y\ (t')$ is given by

$$\left\langle y\left(t'\right)\right\rangle = \int_{-\infty}^{t'} exp\left\{-\left(t'-u\right)\right\} b\cos\left(\omega' u\right)du$$

$$= \frac{b}{1+\omega'}\left(\omega'\sin\omega'\ t' + \cos\omega'\ t'\right) \qquad \text{... (29)}$$

which is the particular part of the solution of the deterministic equation (23), the transient part of the general solution vanishes for large value of the time t.

The mean square fluctuation of $y\,(t')$ is then given by

$$\sigma^2(t') = \left\langle y^2(t')\right\rangle - \left\langle y\,(t')\right\rangle^2 = D \qquad \text{... (30)}$$

The expression shows that the driving force $b \cos \omega' t'$ for the linearized equation (23) has no effect on the dispersion of the growth variable. The fluctuation arises as a result of the random force only. This, infact, follows as a consequence of the Gaussian-Markovinian assumption.

4. Conclusion

The paper aims to study the stochastic behaviour of non-equilibrium fluctuation in logistic non-linear growth of a biological system by the technique of linearization. Two problems have been studied - one is the fluctuation for constant carrying capacity and the other for the periodic carrying capacity of the environment. The paper is concerned with an important biological application of famous 'fluctuaion dissipation theorem' of statistical mechanics. In the first case it is based on a modified form of Nyguist's theorem for quasi - steady state. The result obtained in this case is a modification of the earlier well established and experimentally verified result of population fluctuation for constant carrying capacity. In the second case of periodic carrying capacity the theory is based on the extended form of "fluctuation-dissipation-theorem" for driving system. The results obtained in this case are in good agreement with those of Nisbet and Gurney [5] who justified their results in the light of Jillson's experiment on the culture of Tribolium Castaneum.

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AN EXTENSION OF ANCIENT INDIAN CUBE-ROOT METHOD

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ABSTRACT

In this paper an attempt is made to extend Bhāskarācārya's method of obtaining cube-root of a perfect cubic number to 5th root of any integer. The method indeed may be applied to compute n^{th} root of any positive integer.

1 : INTRODUCTION

Ancient Indian Mathematicians made an attempt to find out the cube-root of perfect numbers. For a detailed discussion refer to Līlāvatī of Bhāskara II (see also Bag [1], Datta and Singh [2], Jha [3], Mishra [7] and Naimpally *et al.* [8]. An attempt has been made by Lal [6] to give a chronological description of the methods for obtaining the real cube root of an integer given by the Hindu Mathematicians. Bhāskara II, in his Līlāvatī, gave a method to find out (real) cube-root of a positive integer. It seems that ancient Indian Pandits knew cube-root techniques much before Bhāskara (b.1114 A.D.). Bhāskara's method is slightly different from his predecessors.

In this paper we make an attempt to give a general method to compute real n^{th} root of an integer. Apart from including several examples to illustrate the technique of computing 5th, 6th, 7th roots of a number,

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'quick computation' is also indicated in certain cases, particularly when the root to be computed is of two digits.

Definition : A number is said to be a perfect number of order n (or simply a perfect number in this paper) if it is expressible in the form of x^n, where n is a positive integer and x is any rational number.

First look at the following table. Calculations at various steps become a little easier if one remembers the message of this table.

TABLE -1

SHOWING END OF ROOTS OF PERFECT NUMBERS ACCORDING TO THEIR UNIT PLACES

Number	Unit places in different powers														
	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	6	2	4	8	6	2	4	8	6	2	4	8
3	3	9	7	1	3	9	7	1	3	9	7	1	3	9	7
4	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	9	3	1	7	9	3	1	7	9	3	1	7	9	3
8	8	4	2	6	8	4	2	6	8	4	2	6	8	4	2
9	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9

Note that :

1. 5th, 9th, 13th, 17th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 2, 3, 4, 5, 6, 7, 8 & 9.
2. 6th, 10th, 14th, 18th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 4, 9, 6, 5, 6, 9, 4 & 1.

3. 7th, 11th, 15th, 19th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 8, 7, 4, 5, 6, 3, 2, & 9.

4. 8th, 12th, 16th, 20th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 6, 1, 6, 5, 6, 1, 6 & 1.

5TH ROOT OF A POSITIVE INTEGER

Note that fifth power of number 1 through 9 are respectively 1, 32, 243, 1024, 3125, 7776, 16807, 32768 and 59049

Given a perfect positive integer of order 5, to find out its fifth positive real root.

Step I : Put a dash on the unit place and a dot on each of the next four digits. Repeat this process till the last digit of the number. Note that the number of dashes equals the number of digits (before decimal) in the proposed fifth root. For example, if the given number consists of 13 digits, then it is divided by the "dash-dot" method into 3 blocks and the first two blocks consist of 5 digits each and the last block consists of only 3 digits.

Step II : From the last block subtract the 5^{th} power of the greatest possible positive integer, say a.

Step III : Associate with this remainder, the first digit of the next block. Now divide this new number by $5a^4$. Let the quotient be b. Evidently $0 \leq b \leq 9$ (see Example 2.1 wherein 4 is associated with the remainder 111 to give a new number 1114, and $b = 7$). This new number may be called 2nd Pada (or Pada II), while the (last) block with which we started the actual computation may be called "first Pada" (or simply Pada I).

Step IV : Form Pada III by taking down this new remainder and the next digit. (For example, our third Pada in Example 2.1 is 5548). Subtract $10a^3b^2$ from Pada III, $10a^2b^3$ from Pada IV, $5ab^4$ from Pada V and b^5 from Pada VI. (If subtraction of $10a^3b^2$ is not possible, reduce b of Step III until subtraction becomes possible, and now the subtraction is done with the reduced b).

Step V : If more blocks are left out divide Pada VII by $5(10a+b)^4$ to get quotient c (where $c \leq 9$). Subtract $10\,(10a+b)^3c^2$ from Pada VIII, $10\,(10a+b)^2c^3$ from Pada IX, $5(10a+b)c^4$ from Pada X and c^5 from Pada XI respectively. (If subtraction of $10\,(10a+b)^3c^2$ is not possible, c is reduced and adjusted in the same way as b).

Step VI : If more blocks are still left out, divide the next Pada XII by $5\,(100a+10b+c)^4$ to get quotient d where d (≤ 9) is adjusted, if necessary, in the same way as b and c). Subtract $10\,(100a+10b+c)^3d^2$, $10(100a+10b+c)^2d^3$, $5\,(100a+10b+c)d^4$ and d^5 from the next Padas respectively.

The process may be extended in this fashion if more blocks are still left out.

Results : (i) For the number having two blocks root is 10a+b, (where b is at the unit place for perfect numbers only).

(ii) For a number having three blocks (respectively four blocks), root is 100a+10b+c (respectively 1000a+100b+ 10c+d) wherein c (respectively d) is at the unit place.

Now we illustrate the method by examples.

Example *2.1* Find the 5th real root of 14348907.

Computation may be done as follows wherein a and b assumed in Steps II-V are given in this and subsequent examples only for the sake of clarity.

$$\dot{1}\;\dot{4}\;\overset{'}{\dot{3}}\;\dot{4}\;\dot{8}\;\dot{9}\;\dot{0}\;\overset{'}{\dot{7}}$$

$$\begin{array}{llrl}
a^5 = 2^5 & & & 2 = a \\
 & & 32 & \\
5a^4 = 5.2^4 & = & 80\overline{)\,1114}\;(& 7 = b \\
5a^4b = 560 & & \underline{560} & \\
10a^3\,b^2 = 10.2^3\,.7^2 & & 5548 & \\
\quad = 3920 & & \underline{3920} & \\
10a^2\,b^3 = 10.2^2.7^3 & & 16289 & \\
\quad = 13720 & & \underline{13720} & \\
 & & 25690 &
\end{array}$$

$$\begin{array}{ll}
5ab^4 = 5.2.7^4 & \\
= 24010 & \underline{24010} \\
 & 16807 \\
b^5 = 7^5 & \\
= 16807 & \underline{16807} \\
 & \times
\end{array}$$

Hence 5th root of 14348907 is 27.

It is important to note that, in view of the observations made on Table 1,7 seems to be at the unit place of the proposd 5th root. Since there are two blocks, the answer will comprise of only two digits. Consequently the Calculations beyond Pada II are not needed, if we know that the given number is a perfect number of 5th order.

Example 2.2 Find 5th root of 4747561509943.

We have the following unnarrated steps showing the details of calculations.

$$\begin{array}{lll}
 & \dot{4}\dot{7}\,\acute{4}\,\dot{7}\dot{5}\dot{6}\dot{1}\,5\,'\,\dot{0}9\dot{9}\dot{4}\,\dot{3} & \\
a^5 = 3^5 & & 3 = a \\
 & \quad 243 & \\
5a^4 = 5.3^4 = & 405\overline{)\,2317}\,(\,4 = b & \\
10\,a^3b^2 = 10.3^2.4^2 & \underline{1620} & \\
 & 6975 & \\
= 4320 & \underline{4320} & \\
10\,a^2b^3 = 10.3^2.4^3 & 26556 & \\
= 5760 & \underline{5760} & \\
5ab^4 = 5.3.4^4 & 207961 & \\
= 3840 & \underline{3840} & \\
 & 2041215 & \\
b^5 = 4^5 = 1024 & 1024 & \\
5(10a+b)^4 = 5\,(34)^4 = 6681680 & \overline{)\,20401910}\,(\,3 = c & \\
10(10a+b)^3\,c^2 = 10(34)^3\,.3^2 & \underline{20045040} & \\
 & 3568709 & \\
= 3537360 & \underline{3537360} & \\
10\,(10s+b)^2\,c^3 = 10(34)^2\,.3^3 & 313499 & \\
= 312120 & \underline{312120} & \\
5(10a+b)\,c^4 = 5(34)\,.\,3^4 & 13794 & \\
= 13770 & \underline{13770} & \\
 & 2'43 &
\end{array}$$

$$\begin{array}{ll} c^5 = 3^5 = 243 & \qquad\qquad \underline{243} \\ & \qquad\qquad \times \end{array}$$

Hence 5th root of 4747561509943 is 343.

Example 2.3 : Find 5th root of 14693280768.

$$\overset{'}{1}\ \dot{4}\,\dot{6}\,\dot{9}\,\dot{3}\,\dot{\overset{'}{2}}\ \dot{8}\,\dot{0}\,\dot{7}\,\dot{6}\,\dot{\overset{'}{8}}$$

$$\begin{array}{lll} a^5 = 1^5 & & \qquad 1 = a \\ & \overset{1}{5\overline{)46}} & \\ 5a^4 = 5.1^4 = & & \qquad (08 = b \\ 10a^3\,b^2 = 10.1^3\,.\,(08)^2 & \underline{40} & \\ & 693 & \\ \quad = 640 & \underline{640} & \\ 10a^2\,b^3 = 10\,(1)^2\,(08)^3 & 5328 & \\ \quad = 5120 & \underline{5120} & \\ 5ab^4 = 5.1.\,(08)^4 & 20807 & \\ \quad = 20480 & \underline{20480} & \\ b^5 = (08)^5 & 32768 & \\ \quad = 32768 & \underline{32768} & \\ & \times & \end{array}$$

Here 4 is not divisible by 5 and so we put 0 as quotiant and bring down the next digit 6. Note that 46 is our Pada II.

Hence 5th root of 14693280768 is 108.

3 : 5th ROOT METHOD FOR NON-PERFECT NUMBERS

Step I : Put dashes and dots as mentioned earlier to subdivide the number into blocks of five.

Step II : Put decimal and zeros after the unit place.

Example 3.1 : Find 5th root of 2.

We do it as before.

$a^5 = 1^5$ $\quad$ 2.00000

1

$5a^4 = 5.1^4 =$ $\quad$ $5\overline{)\,10\,}($ $1=b$ $\quad$ $1 = a$

$\underline{5}$

50

$10a^3\,b^2 = 10.\,(1)^3.\,(1)^2$ $\quad$ $\underline{10}$

$= 10$

400

$10a^2\,b^3 = 10.1^2.1^3 = 10$ $\quad$ $\underline{10}$

3900

$5ab^2 = 5.1.1^4 = 5$ $\quad$ $\underline{5}$

38950

$b^5 = 1^5 = 1$ $\quad$ 1

$5\,(10a + b)^4 = 5.\,(11)^4 = 73205$ $\quad$ $\overline{)389490\,}($ $5=c$

$\underline{366025}$

23465 (Remainder)

Hence 5^{th} root of 2 (till two places of decimal) is 1.15.

Example 3.2 : Find 5th root of 68949

We follow again the previous method.

$\dot{6}\dot{8}9\dot{4}\,\dot{9}'\ .000000$

$a^5 = 9^5$ $\quad$ 5 9 0 4 9 $\quad$ $9 = a$

$5a^4 = 5.9^4 =$ $\quad$ 32805 $\overline{)\,99000\,}($ $2=b$

$\underline{65610}$

333900

$10a^3\,b^2 = 10.9^3.2^2 = 29160$ $\quad$ $\underline{29160}$

3047400

$10a^2b^3 = 10.9^2.2^3 = 6480$ $\quad$ $\underline{6480}$

30409200

$5ab^4 = 5.9.2^4 = 720$ $\quad$ $\underline{720}$

304084800

$b^5 = 2^5 = 32$ $\quad$ 32

$$\begin{array}{lrl}
5\,(10a+b)^4 = 5(92)^4 = & 177065680\overline{)3040847680}\,(& 9 = c \\
 & \underline{1593591120} & \\
 & 14472565600 & \text{(Remainder)}
\end{array}$$

We were looking for 9.29 (correct to two places of decimal).

Example 3.3 : Find 5th root of 34867. 84401.
(Given that this is a perfect number of order 5.)

$$\begin{array}{lrl}
 & \dot{3}\dot{4}\dot{8}\dot{6}7.84401 & \\
a^5 = 8^5 & 32768 & 8 = a \\
5a^4 = 5.8^4 = & 20480\overline{)20998}\,(& 1 = b \\
10a^3\,b^2 = 10.\,8^3\,.\,1^2 & \underline{20480} & \\
 & 5184 & \\
= 5120 & \underline{5120} & \\
10a^2\,b^3 = 10.\,8^2\,.\,1^3 & 644 & \\
= 640 & \underline{640} & \\
5ab^4 = 5.\,8\,.\,1^4 & 40 & \\
= 40 & \underline{40} & \\
 & 1 & \\
b^5 = 1^5 & \underline{1} & \\
 & \times &
\end{array}$$

Hence 5th root of 34867. 84401 is 8.1

4 : 6TH ROOT

Example 4.1 : Find 6th root of 191102976.

$$\begin{array}{lrl}
 & \dot{1}\dot{9}\;\dot{1}\;\dot{1}\dot{0}\dot{2}\dot{9}\dot{7}\,\dot{6} & \\
a^6 = 2^6 & 64 & \\
6a^5 = 6.2^5 = & 192\overline{)1271}\,(& 4 = b \\
 & \underline{768} & \\
15a^4\,b^2 = 15.2^4.4^2 & 5030 & \\
= 3840 & \underline{3840} & \\
 & 11902 & \\
20a^3b^3 = 20.2^3.4^3 = 10240 & \underline{10240} & \\
 & 16629 & \\
15a^2\,b^4 = 15.2^2.4^4 = 15360 & \underline{15360} & \\
 & 12697 &
\end{array}$$

$6a.b^5 = 6.2.4^5 = 12288$ $\quad \underline{12288}$

4096

$b^6 = 4^6 = 4096$ $\quad \underline{4096}$

×

Hence 6th root of 191102976 is 24.

Example 4.2 : Find 6th root of 17685.

17685.000000

$a^6 = 5^6$ $\quad$ 15625 $\quad 5 = a$

$6a^5 = 6.5^5 = 18750$ $\quad 18750\overline{)\,20600}\ (1 = b$

$\underline{18750}$

18500

$15a^4b^2 = 1^5.5^4.1^2 = 9375$ $\quad \underline{9375}$

91250

$20\,a^3b^3 = 20.5^2.1^3 = 2500$ $\quad \underline{2500}$

887500

$15a^2b^4 = 15.5^2.1^4 = 375$ $\quad \underline{375}$

8871250

$6\,a.b^5 = 6.5.1^5 = 30$ $\quad \underline{30}$

88712200

$b^6 = 1^6 = 1$ $\quad$ 1

$6\,(10a + b)^5 = 6\,(51)^5 =$ $\quad 1274551506\overline{)\,8871219900}\ (\ 06 = c$

$\underline{7647309036}$ (Remainder)

1223910864

Hence 6th root of 17685 is 5.106

5 : 7TH ROOT

Example 5.1 : Find 7th root of 170859375

1 7 0 8 5 9 3 7 5

$a^7 = 1^7$ $\quad$ 1 $\quad 1 = a$

$7a^6 = 7.1^6 = 7$ $\quad 7\overline{)\,160}\ (\ 5 = b$

$\underline{35}$

1258

$21\,a^5.b^2 = 21.1^5.5^2 = 525$ $\quad \underline{525}$

7335

$35a^4b^3 = 3^5.\ 1^4\ .\ 5^3 = 4375$ $\quad \underline{4375}$

$$\begin{array}{lr}
 & 29609 \\
35a^3b^4 = 35 \,.\, 1^3 \,.\, 5^4 = 21875 & \underline{21875} \\
 & 77343 \\
21\, a^2b^5 = 21 \,.\, 1^2 \,.\, 5^5 = 65625 & \underline{65625} \\
 & 117187 \\
7ab^6 = 7.\, 1 \,.\, 5^6 \; = 109375 & \underline{109375} \\
 & 78125 \\
b^7 = 5^7 = 78125 & \underline{78125} \\
 & \times
\end{array}$$

Hence 7th root of 170859375 is 15.

n^{th} ROOT

Methods discussed above may be extended to obtaining n^{th} root (wherein n is any positive integer) of a number, say N. Subdivide the digits of N into blocks of n digits by putting a dash on the digit at the first place (i.e. at the unit place) and a dot on each of the next (n-1) digits. Continue this process. The last block, which is our Pada I, might not consist of n digits. Now subtract from the Pada I the n^{th} power of the greatest possible digit. Now form Pada II by associating the first digit of the last but one block with the remainder. Now divide Pada II by $na^{n-1} = {}^nC_1a^{n-1}$* . Assume that the quotient is b. form Pada III by associating the next digit of the last but one block with the remainder obtained on dividing by na^{n-1}. Now subtract from Pada III ${}^nC_2a^{n-1}b$. Continue this process of subtraction from the subsequent Padas up to ${}^nC_nb^n$. If the digits of N have not been exhausted, continue the whole process starting with $n(10a+b)^{n-1}$ instead of a . See Example 2.2.

Indeed the method is some what an inversion of the Binomial Theorem:

$$(a+b)^3 = a^3 + 3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+ 4a^3b+6a^2 b^2+4ab^3 +b^4$$

$$(a+b)^5 = a^5+5a^4 b+10a^3 b^2 +10a^2 b^3 + 5 ab^4+ b^5$$

---

* ${}^nC_r$ stands for n! / r ! (n– r) !

……………………………………………………

$$(a+b)^{n}=a^{n} + {}^{n}C_{1}a^{n-1}b+ {}^{n}C\,a^{n-2}b^{2}+ .... + {}^{n}C_{n}+b^{n}$$

In order to make the 'inversion process' more tangible, we obtain below 6th root of an algebraic expression. For the sake of simplicity we find out 6th root of ($a^6+ 6a^5 b+15a^4 b^2+20a^3 b^3+15a^2 b^4+6ab^5+ b^6$). The following is the calculation chart:

$$a^6 + 6a^5 b+15a^4 b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$= a$$

$$\begin{array}{ll}
6a^5 & \quad a^6 \\
 & \left.\right)\overline{6a^5b}\,(\,b \\
 & \quad \underline{6a^5b} \\
 & \qquad 15a^4b^2 \\
15a^4b^2 & \qquad \underline{15a^4b^2} \\
 & \qquad\quad 20a^3b^3 \\
20a^3b^3 & \qquad\quad \underline{20a^3b^3} \\
 & \qquad\qquad 15a^3b^4 \\
15a^2b^4 & \qquad\qquad \underline{15a^2b^4} \\
 & \qquad\qquad\quad 6ab^5 \\
6ab^5 & \qquad\qquad\quad \underline{6ab^5} \\
 & \qquad\qquad\qquad b^6 \\
b^6 & \qquad\qquad\qquad \underline{b^6} \\
 & \qquad\qquad\qquad \times
\end{array}$$

($a^6+ 6a^5 b+15a^4 b^2+20a^3 b^3+15a^2 b^4+6ab^5 +b^6$)1/6 = a+b

Note that in this algebraic expression 6th root will be a+b and not 10a+b since a,b are not restricted to digits only.

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PRESERVATION OF GENERALIZED CONTINUITY

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ABSTRACT

The main purpose of this paper is to study the effects of uniform convergence and its various generalizations on functions which are nearly-continuous or quasi-continuous. It is known that uniform convergence preserves near continuity as well as quasi-continuity. In this paper a general result is proved to the effect that uniform convergence preserves several weaker forms of continuity including near and quasi continuity. Investigations are also made concerning Dini convergence, Leader convergence, Simple uniform convergence etc. in relation to functions which are nearly or quasi continuous. Necessary tools in the form of pretopology and preproximity are also developed.

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KEY WORDS AND PHRASES

Uniform convergence, Arzelà convergence, Dini convergence, Leader convergence, Quasi-uniform convergence, Simple uniform convergence, Simple Leader convergence, Semi-R_0, Semicompact, Pretopology, Preproximity, Near c** ontinuity, Quasi-continuity, Correct uniformity.

1. INTRODUCTION

Among the various generalizations of continuity, besides upper and lowr semicontinuity, two stand out viz.. *near continuity* (which Blumberg [3] called *densely approached*) and *quasi-continuity* (Kempisty [9]). Near continuity occurs naturally in problems involving open mapping and closed graph theorems of Functional Analysis [13]. On the other hand quasi-continuity is at the basis of results involving joint versus separate continuity. Hence it is no wonder that these two generalizations of continuity have been rediscovered several times under differenent names! In this paper we propose to study the effects of various convergences, which are generalizations of uniform convergence, on these two weak forms of continuity. For a study of continuous functions the reader is referred to (Di Concilio-Naimpally [4]).

Suppose (X, τ) and (Y, σ) are nonempty topological spaces. When we wish to discuss uniform convergence or generalisations we naturally assume that σ is Tychonoff and has a compatible (generalized) uniformity as well as a compatible EF proximity δ_2.

(1.1) *Definition* : A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is **quasi continuous** at $x_0 \in X$ iff for each $U \in \tau$, $V \in \sigma$, $x_0 \in U$, $f(x_0) \in V$ there is a nonempty set $W \in \tau$ such that $W \subset U$ and $f(W) \subset V$. Clearly f is quasi continuous iff for each $V \in \sigma$, $f^{-1}(V) \subset f^{-1}(V)^{0-}$. where 0 and - are interior and closure operators respectively (Kempisty [19]).

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(1.2) *Definition* : A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is **nearly continuous** at $x_0 \in X$ iff for each $V \in \sigma$ containing $f(x_0)$, there is a nbhd. U of x_0 such that $U \subset f^{-1}(V)^-$. Obviously f is nearly continuous iff for each $V \in \sigma$, $f^{-1}(V) f^{-1}(V)^{-0}$.

Although there is a vast literature on quasi continuity and near continuity, it is not widely known that they together characterize continuity when (Y, σ) is regular.

(1.3) *Theorem.* (Garg-Naimpally [6]) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi continuous and nearly continuous then f is θ-continuous. Hence if (Y, σ) is regular, then f is continuous if and only if f is quasi-continuous and nearly continuous

Now we show how quasi continuous and nearly continuous functions can be considered as continuous when the topology τ on the domain X is replaced by larger families called *pretopologies*. From the definitions (1.1) and (1.2) it is clear that the following families are associated with quasi continuous and nearly continuous functions respectively :

(1.4) $$\tau_q = \{A \subset X : A \subset A^{0-}\}$$

(1.5) $$\tau_n = \{A \subset X : A \subset A^{-0}\}$$

It is easy to check that τ_q and τ_n both contain τ and are closed under *arbitrary unions* but not necessarily under *finite intersections*. This leads us to the following definition :

(1.6) *Definition* : A family of subsets ρ of X is a **pretopology** iff ρ contains ϕ, X and is closed under arbitrary unions. (X, ρ) is called a **pretopological space.**

(1.7) *Remarks*. It is now obvious that

(a) $f : (X, \tau) \to (Y, \sigma)$ is quasi continuous iff $f : (X, \tau_q) \to (Y, \sigma)$ is continuous.

(b) $f : (X, \tau) \to (Y, \sigma)$ is nearly continuous iff $f : (X, \tau_n) \to (Y, \sigma)$ is continuous.

We now show that uniform convergence preserves continuity even when the domain space is pretopological. This result includes, as special cases, the results of Bledsoe [2] concerning neighborly i.e. quasi-continuous functions and Husain [8] concerning almost i.e. nearly continuous functions.

(1.8) *Theorem.* (Garg-Naimpally) Suppose (X, ρ) is a pretopological space and (Y, ν) an AN uniform space. If a net of continuous functions $\{f_n : n \in D\}$ converges uniformly to a function f, then f in continuous.

Proof. (We may suppose that all entourages are symmetric in this and all subsequent proofs.) For $V \in \nu$ there is a $V_1 \in \nu$ such that $V_1^3 \subset V$. Since $f_n \to f$, there is an $m \in D$ such that for all $n > m$ and for each $x \in X$, $f_n(x) \in V_1 [f(x)]$. Since each f_n is continuous at $x_0 \in X$, there is $U \in \tau$ such that $x_0 \in U$ and $f_n(U) \subset V_1 [f_n(x_0)]$ where we may suppose that $n > m$ then $f(U) \subset V_1^3 [f(x_0)] \subset V\{fx_0)]$. Hence f is continous at x_0.

(1.9) *Corollary.* Uniform convergence preserves quasi-continuity (Bledsoe [2]) and near continuity (Husain [8]).

2. PRETOPOLOGY AND PREPROXIMITY.

If (X, ρ) is a pretopological space, then a set C is *closed* iff $X - C \in \rho$. We now define a *preclosure operator* $\mathrm{pcl} : P(X) \to P(X)$:

(2.1) *Definition*: $\mathrm{pcl}(A) = \cap \{C : A \subset C, X - C \in \rho\}$.

The following result is obvious :

(2.2) *Theorem* : The preclousre opertaor pcl on a pretopological space (X, ρ) satisfies:

(a) $\mathrm{pcl}(\phi) = \phi$,

(b) $A \subset \mathrm{pcl}(A)$ for each $A \subset X$,

(c) $\mathrm{pcl}(\mathrm{pcl}(A)) = \mathrm{pcl}(A)$,

(d) $A \subset B$ implies $\mathrm{pcl}(A) \subset \mathrm{pcl}(B)$.

(2.3) *Remarks*. Of course, pcl does not satisfy the union axiom but it satisfies the weaker axiom (d) which is equivalent to (d') $\mathrm{pcl}(A) \cup \mathrm{pcl}(B) \subset \mathrm{pcl}(A \cup B)$.

The next result is easy to prove.

(2.4) *Lemma*. Let (X, ρ) be a pretopological space. Then, for each $A \subset X$, $\mathrm{pcl}(A) = \{x \in X : \text{for each } U \in \rho,\ x \in U \Rightarrow U \cap A \neq \phi\}$

Furthermore, every preclosure oeprator pcl satisfying 2.2 (a) - (d), induces a pretopology ρ' where

$\rho' = \{U : \mathrm{pcl}(X - U) = X - U\}$.

Notation. $\underline{A} = \mathrm{pcl}(A)$.

We now define a preproximity.

(2.5) *Definition :* A binary relation $\not\delta$ on $P(X)$ is called a **basic preproximity** iff it satisfies.

(a) $A \not\delta B$ implies $B \not\delta A$,

(b) $A \not\delta B$ implies $A \neq \phi$, $B \neq \phi$,

(c) $A \cap B \neq \phi$ implies $A \not\delta B$,

(d) $A \not\delta B$ and $B \subset C$ implies $A \not\delta C$

A basic preproximity $\not\delta$ is called an **S-preproximity** if it satisfies :

(S) $a \,\delta\, B$ and $b \not\delta C$ for each $b \in B$, then $a \not\delta C$.

A basic preproximity δ is a **LO-Preproximity** if it satisfies:

(LO) $A \delta B$ and $b \delta C$ for each $b \in B$ implies $A \delta C$.

A basic preproximity is an **R-preproximity** if it satisfies :

(R) $a \delta B$ implies there is an $E \subset X$ such that $a \delta E$ and $(X - E) \delta B$.

(2.6) *Theorem* : The basic preproximity δ on X induces an operator δ on P(X) to P(X):

(2.7) $A^{\delta} = \{ x \in X : x \delta A \}$

The above defines a preclosure operator if an only if δ is an S-preproximity. The pretopology induced by a preproximity δ is denoted by $\rho(\delta)$. We say that δ and $\rho(\delta)$ are **compatible.**

(2.8) *Definition* : A pretopological space (X, ρ) is **(symmetric)** R_0 iff $x \in \underline{y}$ implies $y \in \underline{x}$.

The following results are proven in a similar manner to the corresponding results on proximities and topological spaces [11].

(2.9) *Theorem*. Every separated LO preproximity space (X, δ) induces a T_1 pretopological space $(X, \rho(\delta))$. Conversely, every T_1 pretopological space (X, ρ) has the finest compatible LO- preproximity δ_0 as well as the coarsest compatible LO- preproximity δ_{CL} where

(2.10) $A \delta_0 B$ iff $\underline{A} \cap \underline{B} \neq \phi$,

(2.11) $A \delta_{CL} B$ iff $A \delta_0 B$ or both $\underline{A}, \underline{B}$ are infinite.

(2.12) *Theorem*. If δ is a compatible LO-preproximity on a pretopological space (X, ρ), then

$$A \delta B \text{ iff } \underline{A} \delta \underline{B}$$

(2.13) *Definition.* A pretopological space (X, ρ) is **regular** iff for each x not in a preclosed set C, there are $U, V \in \rho$, $x \in U$, $C \subset V$ and $U \cap V = \phi$.

(2.14) *Theorem.* A pretopological space (X, ρ) is regular if and only if it has a compatible LO-R preproximity. The coarsest compatible R-S-preproximity δ_{CS} is given by

(2.15) $A \, \delta_{CS} \, B$ iff $(A \cap \underline{B}) \cup (\underline{A} \cap B) \neq \phi$. or A, B are both infinite.

The finest compatible S preproximity δ_S is given by

(2.16) $A \, \delta_S \, B$ iff $(\underline{A} \cap B) \cup (A \cap \underline{B}) \neq \phi$.

3. **QUASI-CONTINUITY.** In this section we pursue the limits of nets of quasi-continuous functions under various convergences. If (X, τ) is a topological space, then members of τ_q (1.4) are called **semi-open** and the resulting preclosure operator is called scl (**semi-closure operator**). In this section $\underline{A}$ denotes scl (A) and (X, τ) is called **semi - R_0** iff (X, τ_q) is R_0 (2.8). Such a space has compatible LO-preproximities defined by (2.10), (2.11) and if (X, τ_q) is regular then it has compatible S-preproximities defined by (2.15), (2.16).

In this section (X, τ) is a semi-R_0 topological space and δ_1 is any preproximity compatible with (X, τ_q). (Y, σ) dentoes an R_0 topological space with a compatible proximity δ_2. $(f_n : n \in D)$ denotes a net of functions on X to Y.

(3.1) *Definition.* A function $f : (X, \delta_1) \rightarrow (Y, \delta_2)$ is **p-quasi-continuous** iff whenever $A \, \delta_1 \, B$ in X, $f(A) \, \delta_2 \, f(B)$ in Y.

(3.2) *Remarks.* It is easy to see that a p-quasi-continous function is quasi-continuous. The following example shows that the converse is not true. Suppose $X = Y = [-1, 1]$ with the usual topology τ. Let δ_2 be the usual metric proximity on Y and δ_1 the coarsest preproximity δ_{CL} defined by (2.11).

$$f(x) = \begin{cases} 0 & at\ x = 0 \\ \sin 1/x & at\ x \neq 0 \end{cases}$$

Then f is quasi-continuous but not p-quasi-continuous. If $A = \{(n\pi)^{-1} : n \in N\}$, $B = \{[(2n+1)\pi/2]^{-1} : n \in N\}$, then $A\ \delta_1\ B$ but $f(A)\ \delta_2\ f(B)$.

For definitions of various convergences such as R.C., L.C., D.C., S.U.C. Q.U.C., S.L.C. and A.C. please see [4]. Here we merely give examples and state results which are not in [4].

(3.3) *Theorem* : If $f_n \xrightarrow{R.C.} j$ and each f_n is (p-) quasi-continuous, then f is (p-) quasi-continuous.

(3.4) *Example*. Let X = Y = R with the usual topology. For each $x \in X$ and $n \in N$ we set:

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$$f_n(x) = f(x) + n.$$

Then f_n and f are quasi-continuous functions, $f_n \xrightarrow{R.C.} f$ but $f_n \not\xrightarrow{P.C.} f$. Since P.C. does not necessarily preserve quasi-continuity (Levine [10]), it follows that P.C. and R.C. are independent.

(3.5) *Corollary*, X.C. preserves (p-) quasi-continuity.

(3.6) *Theorem* : If δ_2 is EF and $f_n \xrightarrow{L.C.} f$ then $f_n \xrightarrow{X.C.} f$; hence LC. preserves (p-) quasi-continuity.

(3.7) *Example*. Suppose X = Y = R with the usual topology. For each $x \in X$ and $n \in N$ we set :

$$f(x) = \begin{cases} x^2 - 1 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$f_n(x) = \left(f(x) + \frac{1}{n}\right)^2$$

Clearly $f_n \xrightarrow{P.C.} f$ but $f_n \overset{L.C.}{\not\longrightarrow} f$. for $A = N - \{1\}$, $B = \{(n+1/n)^2$ $n \in N]$, $f(A)\ \not\delta_2\ B$ but $f_n(A) \cap B \neq \phi$ for each $n \in N$ and so $f_n(A)\ \delta_2\ B$. δ_2 is the coarsest compatible S-proximity on R, then $f_n \xrightarrow{L.C.} f$. and $\xrightarrow{X.C.} f$. Furthermore, if $\not\delta_2$ is the usual EF proximity,the $f_n \xrightarrow{X.C.} f$.

(3.8) *Example* : Suppose X= Y= [-1, 1] with the usual topoloyg δ the finest compatible preproximity (2.10) on (X, τ_q) and δ_2 the uniqu compatible EF proximity on Y. For each $x \in X$, $n \in N$ we define :

$$f(x) \equiv 0$$

$$f_n(x) = \begin{cases} 0 & x = 0 \\ \dfrac{n \sin(1/x)}{1 + n^2 \sin^2(1/x)} & \text{otherwise} \end{cases}$$

Here $f_n \xrightarrow{P.C.} f$. and $f_n \xrightarrow{R.C.} f$ and so $f_n \xrightarrow{X.C.} f$. But there exists sequence $(x_n) \in X$ such that $\sin(1/x_n) = {}^1/_n$. Set $A = \{x_n\}$, $B = \{1/2\}$; th $f(A)\ \not\delta_2\ B$ but $f_n(A)\ \delta_2\ B$ for each $n \in N$. Thus $f_n \overset{L.C.}{\not\longrightarrow} f$. .

(3.9) *Example* : The example (2.9) (c) of Di Concilio-Naimpally [shows that the sequence

$$f_n(x) = \begin{cases} x + 2/n & x \le 0 \\ x + 1/n & x > 0 \end{cases}$$

is a squence of quasi-continuous functions which L.C. converges to th continuous function f(x) = x in the usual proximity. But if $\not\delta_r$ is the R proximity defined by

(3.10) $\quad A\ \delta_r\ B$ iff $(A \cap B^-) \cup (A^- \cap B) \neq \phi$

then $f_n \overset{X.C.}{\not\longrightarrow} f$. and $f_n \overset{L.C.}{\not\longrightarrow} f$.

(3.11) *Theorem* : If (X, τ) is semi-compact (i. e) every semi-open cover of X admits a finite subcover [14] and $f_n \xrightarrow{S.\,L.\,C.} f$ which is quasi-continuous, then $f_n \xrightarrow{Q.\,U.\,C.} f$.

(3.12) *Theorem* : Suppose X is semi-compact and δ_2 is EF. If $f_n \xrightarrow{P.\,C.} f$ and each f_n and f are quasi-continuous, then $f_n \xrightarrow{A.\,C.} f$.

Proof. Suppose f is quasi-continuous, $p \in X$, $B \subset X$, and $f(p) \not\delta_2 f(B)^-$. Since f is quasi-continuous, $f(\underline{B}) \subset f(B)^-$ and since X is semi-compact we may suppose that B is semi-compact. Since δ_2 is EF, there is an open set $V \subset Y$ such the $f(B) \subset V$ and $f(p) \not\delta_2 V$. Suppose $m \in D$. Since $f_n \xrightarrow{P.\,C.} f$., there exists an $m' > m$ such that $f_n(p) \not\delta_2 V$ for each $n > m'$. Also we have $B \subset f_{n_i}^{-1}(V)$ for each $n > m'$. Since B is semi-compact and each $f_n^{-1}(V)$ is semi-open, $B \subset \bigcup_{i \in I_m} f_{n i}^{-1}(V)$. Consequently,

$$B = \bigcup_{i \in I_m} B_i \text{ where } B_i = B \cap f_{n_i}^{-1}(V).$$

Hence $f_{n i}(p) \not\delta_2 f_{n i}(B_i)$ i.e. $f_n \xrightarrow{A.\,C.} f$.

(3.13) *Theorem* : If X is semi-compact and extremally disconnected, δ_2 if EF, $f_n \xrightarrow{A.\,C.} f$, each f_n is quasi-continuous implies f is quasi-continuous.

Proof. In this case the semi-closure operator (Njastad [12]) is a Kuratowski closure operator and we follow the proof of Di Concilio-Namipally [4].

4. NEAR CONTINUITY : In this section we study the convergence of nearly continuous functions. Since several of the results of the last section go through easily for nearly continuous functions as well, we mostly present counter examples in this section.

If (X, τ) is a topological space, then τ_n as defined by (1.5) consists of nearly open sets [13]. Complements of the members of τ_n are called **nearly closed** and they naturally induce a preclosure operator ncl. We write $\underline{A}$ = ncl (A). Suppose (X, τ) is such that ncl $\{x\} = \{x\}$ for each $x \in X$. Then (X, τ_n) has a compatible LO preproximity δ_0 defined by

$$A \, \delta_0 \, B \text{ iff } \mathrm{ncl}\,(A) \cap \mathrm{ncl}\,(B) \neq \phi$$

If (Y, σ), is a topological space, then $f : (X, \tau) \to (Y, \sigma)$ is nearly continuous iff

$$f\,(\underline{A}) \subset f(A)^{-} \text{ for each } A \subset X.$$

Definition of a p-nearly continuous function is similar to that of a p-quasi continuous function.

4.1*Example.* We now present an example of a nearly continuous function which is not p-nearly continuous. $X = Y = R$ with the usual topology.

$$f\,(x) = \begin{cases} 0 & x \in Q \\ 1 & \textit{otherwise} \end{cases}$$

If δ_{CL} denotes the coarsest compatible LO preproximity on (X, τ_n), then

$A \, \delta_{CL} \, B$ but $f(A) \not\delta_2 f(B)$

Where $A = \{n : n \in N\}$, $B = \{n + \sqrt{2} : n \in N\}$ and δ_2 is the usual metric proximity on Y.

(4.2) *Definition.* A topological space (X, τ) is strongly compact iff every nearly open cover of X has a finite subcover [1].

Results analogous to (3.3) and (3.6) are ture for nearly continuous functions. Analogue of Theorem (3.11) also holds if we replace semi-compact by strongly compact. Similarly semi-compact is to be replaced by strongly compact in (3.12).

(4.3) *Lemma*. (Ganster [5]) Let $X = F \cup G$ be the Hewitt representation of X, where F is closed and resolvable, G is open and hereditarily irresolvable with $F \cap G = \phi$. The following are equivalent,

(a) (X, τ_n) is a topology

(b) G is open and each point of F^0 is nearly open.

Analogue of Theorem (3.13) is true if we replace semi-compact by strongly compact and extremally disconnected by (a) or (b) of Lemma (4.3).

We now present counterexamples :

(4.4) *Example*. X = Y = R with the usual topology.

$$f(x) = \begin{cases} 0 & x \in Q \\ 1 & \text{otherwise} \end{cases}$$

$f_n(x) = f(x) + n.$

Then $f_n \xrightarrow{R.C.} f$ but $f_n \not\xrightarrow{P.C.} f$. Since Husain [8] has shown that P.C. does not preserve near continuity, we have P.C. and R.C. independent.

(4.5) *Example* . X = Y= [-1, 1] with the usual topology.

$f(x) = 0$

$$f_n(x) = \begin{cases} 1/n & x \in X-Q \\ nx/(1+n^2x^2) & \text{otherwise.} \end{cases}$$

Then $f_n \xrightarrow{P.C.} f$., $f_n \xrightarrow{R.C.} f$ but $f_n \not\xrightarrow{L.C.} f$, since if $A = \{1/n : n \in N\}$; $B = \{1/2\}$, $f(A) \not\delta_2 B$ but $f_n(A)\ \delta_2 B$.

(4.6) *Example*. X =Y= R with the usual topology.

$$f(x) = \begin{cases} x^3 & x \in Q \\ -x^3 & \textit{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} \left(x + \frac{1}{n}\right)^3 & x \in Q \\ -\left(x + \frac{1}{n}\right)^3 & \textit{otherwise.} \end{cases}$$

Set $A = \{n \in N - \{1\}\}$, $B = \{n + 1/n)^3 : n \in N - \{1\}\}$. Then $f(A) \not\delta'_2$ B but $f_n(A) \cap B \neq \phi$ for each $n \in N$.Consequently, $f_n \overset{L.C.}{\nrightarrow}$ f. But if $\not\delta'_2$ is the R-proximity defined by (3.10), then $f_n \xrightarrow{L.C.}$ f. However, $f_n \xrightarrow{X.C.}$ f if δ_2 is EF or the above.

(4.7) *Example* . X = Y = R with the usual topology.

$$f(x) = \begin{cases} x & x \in Q \\ 0 & \textit{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} x + 1/n & x \in Q - \cup \{O\} \\ x + 2/n & x \in Q + \\ 0 & \textit{otherwise.} \end{cases}$$

If δ'_2 is the usual metric proximity, then $f_n \xrightarrow{L.C.}$ .f. But if $\not\delta'_2$ is (3.10), then $f_n \nrightarrow$ f in X.C. or L.C.

CONCLUSION : We have already noted that the family of nearly open sets or the family of semi-open sets does not form a topology for R. We recall the R with the usual topology is anti-semicompact i.e. every semicompact subset is finite [14]. Furthermore, no closed interval of R can be strongly compact. Hence we cannot have Arzela-type convergence for quasi-continuous or nearly continuous functions on a closed bounded interval of R.

The techniques developed in this paper can also be used to study the convergence of other generalized forms of continuity viz. *weak continuity, complete continuity, α-continuity, supercontinuity* (for definitions see [7]). In fact it is not difficult to set up, for each of these forms of generalized continuity, an appropriate pretopology and an associated LO-preproximity on the domain. Then we can show that these generalized forms of continuity are preserved under U.C., D.C., L.C. and X. C. It is also not difficult to see that U.C. preserves feeble continuity [7].

However, to attack the problem of preservation of *strong continuity, irresoluteness and $\wp$-Continuity* [7], we are forced to set a pre-topology and a pre-proximity on both the domain and the range. In this manner, we can show that irresoluteness, strong continuity and $\wp$-continuity are preserved only under X.C. and D.C.

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A NOTE ON THE STRESS DISTRIBUTION IN A ROTATING CIRCULAR DISK OF NONHOMOGENEOUS MATERIAL UNDER TRANSIENT SHEARING FORCE

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ABSTRACT

In this note, distribution of stress in a nonhomogeneous annular circular plate rotating with time dependent angular velocity has been found. Numerical results have been presented to study the effect of nonhomogeneity on stress components

INTRODUCTION

The problem of stress distribution in a steadily rotating circular disk has been gaining considerable interest among the research workers for its applications in engineering fields. In literature we find many investigators concentrating their attention on the problem under different conditions. The works of Singh and Puri [1], Samanta [2], Chakravorty and Chaudhuri [3] may be referred in this connection. In the present note my aim is to study the problem assuming the material of the disk as nonhomogeneous in the sense that both the Young's modulus E and the density ρ of the material are functions of position. It is also assumed that the disk is rotating with exponentially decaying time-dependent angular velocity. The problem has been completely solved and the expressions for the stresses have been given in terms of infinite series. Finally, to study the effect of nonhomogeneity on the stresses numerical computations have been done and the results have been graphically exhibitted.

FORMULATION OF THE PROBLEM

We assume that the stresses and displacements do not vary across the thickness of the disk and the lateral surface is free from any stress. Taking the centre of the circular plate as pole and using polar coordinates (r, θ) we find that for reasons of symmetry, the radial displacement U_r and the tangential displacement U_θ are independent of θ.

The equations of motion are

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\,\Omega^2 r=\rho\,\frac{\partial^2 U_r}{\partial t^2} \tag{1}$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}-\rho\,\frac{d\Omega}{dt}r=\rho\,\frac{\partial^2 U_r}{\partial t^2}$$

where σ_r, σ_θ and $\tau_{r\theta}$ are the radial, tangential and the shearing stress components respectively given by

$$\sigma_r=\frac{E}{1-\upsilon^2}\left[\frac{\partial U_r}{\partial r}+\nu\,\frac{U_r}{r}\right]$$

$$\sigma_\theta=\frac{E}{1-\upsilon^2}\left[\upsilon\frac{\partial U_r}{\partial r}+\frac{U_r}{r}\right] \tag{2}$$

$$\tau_{r\theta}=\frac{E}{2(1+\upsilon)}\left[\frac{\partial U_\theta}{\partial r}-\frac{U_\theta}{r}\right]$$

E = E (r) is the Young's modulus, $\rho=\rho(r)$ is the variable density, ν is the Poisson's ratio of the material and Ω = Ω (t) is the angular velocity of the disk. We assume that Ω varies exponentially with time such that

$$\Omega=\Omega_0 e^{-p\tau}$$

where Ω_0 and p (> O) are constants and $\tau \;\; t/t_0$

The boundary conditions are

$$\tau_{r\theta} = -M e^{-pt} \qquad \text{on } r = a \; (M \text{ is a constant})$$

$$= 0 \qquad \text{on } r = b$$

$$\sigma_r = 0 \qquad \text{on } r = a \text{ and } r = b$$

where a and b are the outer and inner radii of the annular disk. The non-homogeneity of the material of the disk is assumed to obey the laws

$$E = E_o e^{\beta x}$$

$$\rho = \rho_o e^{\beta x}$$

where $x = r/b$ and E_0 and ρ_0 are constants and β is a real number.

Substitutions from (2) and (5) into (1), leads to two coupled differential equations in U_r and U_θ. The nature of angular velocity Ω in (3) suggests U_r and U_θ in the following forms

$$U_r = F_1(x)\, e^{-2p\tau/b} \tag{6}$$

$$U_\theta = F_2(x)\, e^{-p\tau/b}$$

With such a choice, the field equations in non-dimensional variable x may be written as

$$\frac{d^2 F_1}{dx^2} + \left(\beta + \frac{1}{x}\right)\frac{dF_1}{dx} + \left(\frac{\upsilon'\beta}{x} - \frac{1}{x^2} - K_2\right)F_1 = -K_1 x \tag{7}$$

and

$$\frac{d^2 F_2}{dx^2} + \left(\beta + \frac{1}{x}\right)\frac{dF_2}{dx} - \left(\frac{\beta}{x} + \frac{1}{x^2} + K_4\right)F = -K_3 x \tag{8}$$

where

$$K_1 = \rho_o \, \Omega_o^2 b^2 (1-\upsilon^2)/E_o$$

$$K_2 = 4 \, \rho_o \, p^2 \, b^2 (1-\upsilon^2)/E_o \, t_o^2$$

$$K_3 = 2 \, \rho_o \, \Omega_o \, b^2 p \, (1+\upsilon)/E_o \, t_o$$

$$K_4 = 2 \, \rho_o \, b^2 p^2 \, (1+\upsilon)/E_o \, t_o^2$$

Thus our problem is to solve (7) and (8) under boundary conditions (4).

Solution of the problem

Setting $F_1 = x. V_1$

and $F_2 = x. V_2$, equations (7) and (8) become

$$x \frac{d^2 V_1}{dx^2} + (3+\beta x) \frac{d V_1}{dx} + \left[\beta \, (1=\upsilon) - K_2 x\right] V_1 = -K_1 x \tag{9}$$

$$x \frac{d^2 V_2}{dx^2} + (3+\beta x) \frac{d V_2}{dx} - (K_4 x) V_2 = -K_3 x \tag{10}$$

The complementary equations of (9) and (10) may be easily transformed into confluent hypergeometric differential equations by suitable substitutions. Since the solution of such equation is known the general solution of (9) (10) may be given by

$$F_1 = x. e^{s_1 x} \left[A_1 \phi_1 \, (C_1 x) + B_1 \, \psi_1 (C_1 x)\right] + x \eta_1 \, (x) \tag{11}$$

and

$$F_2 = x. e^{s_2 x} \left[A_2 \phi_2 \, (C_2 x) + B_2 \, \psi_2 (C_2 x)\right] + x \eta_2 \, (x) \tag{12}$$

Where ϕ_1, ψ_1 and ϕ_2, ψ_2 are respectively known solutions of the confluent hypergeometric equations, [4]

$$\xi \frac{d^2 y}{d\xi^2} + (3-\xi) \frac{dy}{d\xi} - \alpha_1 y = 0$$

and

$$\xi \frac{d^2y}{d\xi^2} + \left(3-\xi\right)\frac{dy}{d\xi} - \alpha_2 y = 0$$

Where

$\alpha_1 = (3s_1 + \beta(1 + \upsilon))] / [2s_1 + \beta], C_1 = -(2s_1 + \beta)$

$\alpha_2 = (3s_2 / [2s_2 + \beta], C_2 = -(2s_2 + \beta)$

s_1 *is a root of* $\theta^2 + \beta\theta - K_2 = 0$, *while* s_2 *is that of* $\theta^2 + \beta\theta - K_4 = 0$

The functions η_1 (x) and η_2 (x) appearing in (11) and (12) are given by

$$\eta_1(x) = \sum_{n=2}^{\infty} a_n x^n$$

$$\eta_2(x) = + K_3 / K_4$$

where

$a_2 = -K_1 / 8$

$a_3 = \beta(3 + \upsilon) . K_1 / 120$

and other a_i's can be calculated from the relation

$(n + 1)(n + 3) a_{n+1} + \beta(1 + \upsilon + n) a_n - K_2 a_{n-1} = 0$ *for* $n \geq 3$.

Knowing F_1 and F_2, we can find U_r and U_θ and U_θ from (6). Using these U_r and U_θ, we get from (2)

$$\sigma_r = \frac{E_0 e^{-2p\tau}}{b\left(1-\upsilon^2\right)}\left[A_1 R_1(x) + B_1 R_2(x) + R_3(x)\right]$$

$$\sigma_r = \frac{E_0 e^{-2p r}}{b\left(1-\upsilon^2\right)}\left[A_1 R_4(x) + B_1 B_5(x) + R_6(x)\right]$$

and

$$\tau_{r\theta} = M . e^{-p\tau}\left[A_2 R_7\langle x\rangle + B_2 R_8(x)\right]$$

.where

$$R_1(x) = [\,(1 + \upsilon + s_1 x)\,\phi_1(C_1 x) + C_1 x\,\phi'_1(C_1 x)\,]\,e^{(\beta + s_1)x}$$

$$R_2(x) = [\,(1 + \upsilon + s_1 x)\,\psi_1(C_1 x) + C_1 x\,\psi'_1(C_1 x)\,]\,e^{(\beta + s_1)x}$$

$$R_3(x) = [\,(1 + \upsilon) + \eta_1(x) + x.\,\eta'_1(x)\,]\,e^{\beta x}$$

$$R_4(x) = [\,(1 + \upsilon + \upsilon x s_1)\,\phi_1(C_1 x) + \upsilon x C_1 .\phi'_1(C_1 x)\,]\,e^{(\beta + s_1)x}$$

$$R_5(x) = [\,(1 + \upsilon + \upsilon x s_1)\,\psi_1(C_1 x) + \upsilon x C_1 .\ \psi'_1(C_1 x)\,]\,e^{(\beta + s_1)}$$
x

$$R_6(x) = [\,(1 + \upsilon)\,\eta_1(x) + x\upsilon\ \eta'_1(x)\,]\,e^{(\beta x)}$$

$$R_7(x) = [\,(x\ s_2)\,\phi_2(C_2 x) + .\,C_2\ \phi'_1(C_2 x)\,]\,e^{(\beta + s_2)x}$$

$$R_8(x) = [\,(x\ s_2)\,\psi_2(C_2 x) + .\,C_2\ \psi'_1(C_2 x)\,]\,e^{(\beta + s_2)x}$$

The constants A_1, B_1, A_2, B_2 *in (13), (14) and (15) may be* determined from the boundary conditions (4), as

$$A_1 = [R_2(1)\,R_3(x_o) - R_2(x_o)\,R_3(1)\,] \,/\, \Delta_1$$

$$B_1 = [R_1(x_o)\,R_3(1) - R_1(1)\,R_3(x_o)\,] \,/\, \Delta_1$$

$$A_2 = [+ R_8(1)\,] \,/\, \Delta_2$$

$$B_2 = - R_7(1)\,] \,/\, \Delta_2$$

where $x_o = a / b$

$$\Delta_1 = R_1(1)\,R_2(x_o) - R_1(x_o)\,R_2(1)$$

$$\Delta_2 = R_7(1)\,R_8(x_o) - R_7(x_o)\,R_8(1)$$

Thus knowing A_1, B_1, A_2, B_2, we get the complete expressions for the components of stresses from (13),(14) and (15).

Numerical results :

To study the nature of stress distribution and also the effect of nonhomogeneity on the stresses we compute radial and tangential stress components along with shearing stress component at different positions in the disk. In our computation we have assumed $p = 1/2, \Omega_0 t_0 = 1, \upsilon = 1/3$ *and* $\rho_0 \Omega^2_0 b^2 = E_0$

We plot variations of

$$P = \left[b\left(1-\upsilon^2\right) e^{2p\tau} \big/ E_0 \right] \sigma_r$$

$$Q = -\left[b\left(1-\upsilon^2\right) e^{2p\tau} \big/ E_0 \right] \sigma_\theta$$

$$R = \left[e^{p\tau} \big/ M \right] \tau_{r\theta}$$

with position in Figure 1, 2 and 3.

The results in the associated homogeneous case ($\beta = 0$) have been shown in broken lines. The effect of non-homogeneity is clear from the graph.

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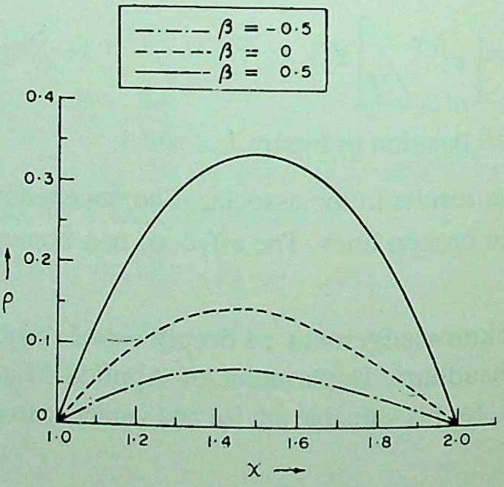

Fig. 1 Variation of radial stress with position (for fixed τ).

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Contents

BIOLOGY

1 G. Prasad, A. Shanker, K.R. Aneja and S.K. Sharma : Effect of temprature on development of cycas ovule rot at Haridwar.

6 V. Shanker, G. Prasad and Shashi Kant Sharma : Effect of Industrial effluent on seed germination of some agricultural Crops.

BIOCHEMISTRY

12 Gopal Chandra : Theory of olfaction/gustation based on dipole moment of molecule.

CHEMISTRY

33 Gopal K. Sinha : A study of essential oil from erigeron linifolius willed.

MATHEMATICS

39 Ljiljana Gajic : On convexity in convex metric spaces with application.

49 P.K. Chaudhari and Subrata Datta : Thermal stress in a semi-infinite rod of nonhomogeneous material.

57 S.L. Singh and B.D. Pant, K.P. Chamola : Coincidices and fixed points of meir-keeler type contractive mappings on menger spaces.

68 C.G. Chakrabarti and R.N. Talapatra : Logistic growth under random driving forces.

78 S.L. Singh and Ramesh Chand : An extension of ancient indian cube-root method.

90 Giuseppe Di Maio and Somashekhar A. Naimapally : Preservation of generalized continuity.

105 Sabyasachi Sen : A note on the stress distribution in a rotating circular disk of nonhomogeneous material under transient shearing force.

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EXTENSIONS OF THE CARISTI-KIRD FIXED POINT THEOREM

Shyam L. Singh*

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ABSTRACT

Two coincidence theorems using an elementary constructive proof are proved in this paper which include *inter alia* a recent substantial generalization of the Caristi-Kirk fixed point theorem due to Bollenbacher and Hicks [1]. Serveral examples provide an insight into the results and raise new questions.

AMS (MOS) subject classifications (1880): 47H10, 54H25.

Keywords and phrases: Coincidence/fixed point theorem, Caristi-Kirk fixed point theorem, weakly preorbitally commuting maps, metric space.

1. INTRODUCTION

The Caristi-Kirk fixed point theorem [2] generalize a number of fixed point theorems and has various proofs ar several applications (see, for instance, [1]-[3], [5 [6]-[8], [10] and the referencesof [8]). However, Caristi' proof, as well as several others, lack iterativ construction of the fixed point. A recent result c Bollenbacher and Hicks [1] (see also Hicks [5]) has a elementary constructive proof and improves the Carist: Kirk theorem substantially.

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In this paper two coincidence theorems (Theorems 1-2) for a pair of maps on an arbitrary set with values in a metric space are given and the result of [1], as well as of others, may be derived as corellaries. Indeed, apart from obtaining the fixed point theorem of Bollenbacher and Hicks [1] exactly as a special case of Theorem 1 (Cf. Remark 1), a variant of their theorem is also obtained as a corellary (Corollary 3) of Theorem 2.

Throughout this paper, let A be an arbitrary set, (X, d) a metric space and f, T maps on A with values in X. A point z in A is said to be a coincidence point of f and T if $fz = Tz$. If for a point $x_o \in A$ there exists a sequence $\{x_n\}$ of points of A such that $fx_{n+1} = Tx_n$ $n=0,1,2....$, then

$$O(T,f;x_0) = \{fx_n : n = 1,2,.....\}$$

is said to be the orbit of (T,f) at x_0. We shall use $O(T,f;x_0)$ as a set and as a sequence as the situation demands. A subspace of X is said to be $(T,f;x_0)$-orbitally complete if the colsure of $O(T,f;x_0)$ is complete. Evidently, the completeness of X implies the orbital completeness, and the space may be $(T,f;x_0)$-orbitally complete without being complete (see, for instance, [11, Ex. 2.4]). A subspace of X will be called (T,f)-orbitally complete f it is $(T,f;x_0)$-orbitally complete for every $x_0 \in A$. $G:X \to (0,\infty)$ is $(T,f;x_0)$-orbitally lower semi-continuous (l.s.c.) if $\{z_n\}$ is a sequence in $O(T,f;x_0)$ and $\lim z_n = p$ implies $G(p) \leq \liminf G(z_n)$. If $A = X$ and $f=I$ (identity map) then G is said to be T-orbitally l.s.c. at x_0; [1], [8,p.62]. If $A = X$ then: (i) T is said to be $(T,f;x_0)$-orbitally continuous [12] if the restriction of T on the closure of $O(T,f;x_0)$ is continuous; (ii) T and f are said to be weakly commuting [9] iff $d(Tfx, fTx) \le d(fx, Tx)$ for every x in **X**; and (iii) T and f will be called weakly x_0-preorbitally commuting (w.x_0-p.c.) iff there exists a positive integer N such that $d(Tfx_n, fTx_n) \le d(fx_n, Tx_n)$ for x_n (in $\ge N$) occurring in $O(T,f;x_0)$. Moreover, T and f will be called (T,f)-orbitally continuous (respectively weakly preorbitally commuting) if they are $(T,f;x_0)$-orbitally continuous (resp. w.x_0-p.c.) for ever $x_0 \in X$. We remark that commutativeity ⇒ weak commutativety ⇒ weak preorbital commutativity but not its vice-versa, (see Example 2, 5 below).

2. RESULTS

Consider the following condition for $\phi : X \to [0,\infty)$ and every y C A :

(CP) $\quad d(fy, Ty) \le \phi(fy) - \phi(Ty)$.

Park [6, Prop. 7] using a non-constructive proof shows the existence of a coincidence point of T and f under this condition when A = X is complete, ϕ l.s.c. and f surjective. The Caristi-Kirk fixed point theorem [2] states that T satisfying (CP) with A = X complete, f = I and ϕ l.s.c. has a fixed point. In the following Coincidence Theorems 1-2, assume that there exists an orbit for (T,f).

THEOREM 1. Let T, f be maps on A with values in X and $\phi : X \to (0,\infty)$. Suppose there exists an $x_0 \in A$ such that (CP) for every $fy \in O(T,f;x_0)$, and f(A) is $(T,f;x_0)$-orbitally complete. Then:

(1.1) $\lim Tx_n = \lim fx_n = fz$ exists for some $z \in A$;

(1.2) $d(Tx_n, fx) \le \phi(Tx_n)$;

(1.3) $Tz = fz$ iff $G(fz) = d(fz, Tz)$ is $(T, f; x_0)$-orbitally l.s.c.;

(1.4) $d(Tx_n, Tx_0) \leq \emptyset(Tx_0)$ and $d(fz, Tx_0) \leq \emptyset(Tx_0)$.

PROOF. Since
$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(fx_{n+1}, Tx_{n+1}) \\ &\leq \emptyset(fx_{n+1}) - \emptyset(Tx_{n+1}) \\ &= \emptyset(Tx_n) - \emptyset(Tx_{n+1}), \end{aligned}$$

that is

$$\sum_{k=0}^{n} d(Tx_k, Tx_{k+1}) \leq \emptyset(Tx_0) - \emptyset(Tx_{n+1}) \leq \emptyset(Tx_0)$$

$\{Tx_n\}$ and $\{fx_n\}$ are Cauchy sequences and both have the same limit. Call it p. Since $p \in f(A)$, there exists a z in A such that $fz = p$. This proves (1.1).

(*) $$\begin{aligned} 0 \leq d(Tx_n, Tx_m) &\leq \sum_{k=n}^{m-1} d(Tx_k, Tx_{k+1}) \\ &\leq \emptyset(Tx_n) - \emptyset(Tx_m) \leq \emptyset(Tx_n). \end{aligned}$$

Making $m \to$ gives (1.2).

The $(T, f; x_0)$-orbital l.s.-continuity of G implies

$$\begin{aligned} 0 \leq d(fz, Tz) = G(fz) &\leq \liminf G(fx_n) \\ &= \liminf d(fx_n, Tx_n) \\ &= \liminf d(fx_n, fx_{n+1}) = 0. \end{aligned}$$

So $fz = Tz$.

If $Tz = fz$ and $\{fx_n\} \subset O(T, f; x_0)$ with $fx_n \to fz$ then $G(fz) = d(fz, Tz) = 0 \leq \liminf d(fx_n, fx_{n+1}) = \liminf d(fx_n, Tx_n) = \liminf G(fx_n)$. This completes the proof of (1.3).

In view of the ineuality (*),

$$d(Tx_0, Tx_n) \leq \emptyset(Tx_0) - \emptyset(Tx_n) \leq \emptyset(Tx_0).$$

Making $n \to \infty$ gives $d(Tx_0, fz) \leq \phi(Tx_0)$. This ends the proof.

THEOREM 2. Let T, f be self-maps of X and $\phi: X \to [0, \infty)$. Suppose there exists an $x_0 \in X$ such that (CP) with A = X holds for every $fy \in O(T, f; x_0)$, and X is $(T, f; x_0)$-orbitally complete. Then :

(2.1) $\lim Tx_n = \lim fx_n = p$ exists, $p \in X$;

(2.2) $d(Tx_n, p) \leq \emptyset(Tx_n)$;

(2.3) if {T, f} is a weakly x_0-preorbitally continuous pair and T, f are $(T, f; x_0)$-orbitally continuous then $T_p = fp$.

(2.4) $d(Tx_n, Tx_0) \leq \emptyset(Tx_0)$ and $d(p, Tx_0) \leq \emptyset(Tx_0)$.

PROORF. In view of the proof of Theorem 1 we need show only (2.3). Since $fx_n \to p$, $Tx_n \to p$, the w.x_0-p. commutativity of T and f implies $\lim d(Tfx_n, fTx_n) = 0$. This together with the orbital continuity of T and f gives $d(Tp, fp) = 0$, i.e. $Tp = fp$.

REMARKS 1. Theorem 3 of Bollenbacher and Hicks [1] generalizing the Caristi-Kirk fixed point theorem is obtained from Theorem 1 when A = X and f = I.

2. If $T(A) \subset f(A)$ then for any $x \in A$ there exists a $y \in A$ such that fy=Tx. Therefore $T(A) \subset f(A)$ implies the existence of an orbit $O(T, f; x_o)$ for every $x_0 \in A$. If A = X and T = I then $O(I, f; x_0) = \{x_n = fx_{n+1}: n = 0, 1, 2, \ldots\}$.

3. If, instead of f(A), T(A) is assumed $(T, f; x_0)$-orbitally complete in Theorem 1 then, as evident from the proof of Theorem 1, fz and Tz are just interchanged in (1.1) - (1.4).

In view of Remark 2, we have the following corollaries from Theorems 1-2.

COROLLARY 1. Let T, f be maps on A with values in X such that $T(A) \subset f(A)$, f(A) a (T,f)-orbitally complete subspace of X and (CP) for everyy $y \in A$. Then for any $x_0 \in A$, there exists a sequence $\{x_n\}$ in A such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots.$, and the conclusions (1.1) - (1.4) are ture.

COROLLARY 2. Let X be a metric space and T, f self-maps of X such that $T(X) \subset f(X)$, X is (T, f)-orbitally complete and (CP) with A = X holds for ever $y \in X$. Then, for any $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, and the conclusions (2.1) - (2.2), (2.4) and the following are true :

(2.3′) if {T, f} is a weakly preorbitally commuting pair and T, f are (T,f)-orbitally continuous then Tp=fp.

Taking T = I in Theorem 2 we have the following fixed point theorem.

COROLLARY 3. Let $f: X \to X$ and $\phi : X \to [0,\infty)$. Suppose there exists an orbit $O(I,f;x_0)$ for some $x_0 \in X$ such that

(P) $d(fy, y) \leq \phi(fy) - \phi(y)$

for ever $y \in O(I,f;x_0)$, and X is $(I,f;x_0)$-orbitally complete Then :

(2.1c) $\lim x_n = p$ exists, $p \in X$;

(2.2c) $d(x_n,p) \leq \phi(x_n)$;

(2.3c) fp = p if f is $(I,f;x_0)$-orbitally continuous, that is p is a fixed point of f if it is continuous at p;

(2.4c) $d(x_n, x_0) \leq \phi(x_0)$ and $d(p,x_0) \leq \phi(x_0)$.

REMARK 4. Corollary 3 is a variant of [1, Th. 3] and an improved version of a result of Park [6, Prop.9]. It is noted in [6,p. 149] that the condition (P) includes a number of contractive type conditions for maps on X. Examples (see below) show that the range of applicability of our results is wider than *inter alia* Goebel's coincidence theorem (Theorem 3 below). He proved it using the Banach contraction principle, while, as also noted in [1, Remark 2], our results include many generalizaitons of Banach's fixed point theorem.

THEOREM 3[4]. Let T,f be maps on A with values in X such that $T(A) \subset f(A)$ and

$$(3.1) \qquad d(Tx, Ty) \leq q\, d(fx, fy)$$

for all $x, y \in A$ and some $q \in (0, 1)$. If f(A) is a complete subspace of X then T and f have a coincidence point in A.

3. EXAMPLES

EXAMPLE 1. This example illustrates Theorem 1 and shows that Theorem 3 is not applicable. Let $A = \{a,b,c\}$; $X = \{1,2,3\}$; $Ta = Tb = 1$, $Tc = 2$; $fa = 1$, $fb = 2$, $fc = 3$; $\emptyset(1) = 5/2$, $\emptyset(2) = 4$, $\emptyset(3) = 6$; and d a metric on X such that $d(1, 2) = d(1,3) = 3/2$, $d(2, 3) = 2$. Then evidently $T(A) \subset f(A)$, $d(fx, Tx) \leq \emptyset(fx) - \emptyset(Tx)$ for every x in A, and a is a coincidence point of T and f. However, Theorem 3 is not applicalbe as T and f do not satisfy (3.1), since $d(Ta, Tc) = d(fa, fc)$.

EXAMPLE 2. Let X be the set of non-negative rationals, d the absolute value metric on X and T, f; $X \rightarrow X$ such that

$Tx = \begin{cases} 2x^2, & x \leq 4 \\ x, & x > 4 \end{cases}$ and $fx = \begin{cases} 4x, & x \leq 1 \\ 8x, & x > 1. \end{cases}$ Let

$$\emptyset(x) = \begin{cases} 1 + x, & x \leq 4 \\ 2 + x, & x > 4. \end{cases}$$

Then $T(X) \not\subseteq f(x)$ and $d(fx, Tx) \leq \phi(fx) - \phi(TX)$ for every x in X. Moreover, for $x_0 = 1$,

$$x_n = 2^{-2^n+1},\ n = 1, 2, 3, \ldots, \text{ and}$$

$$O(T,f;x_0) = \{2^{-2^n+3} : n = 1, 2, 3, \ldots\} \text{ converges to } 0.$$

Evidently, Theorem 1 applies. Theorem 2 also applies as it is easy to check that $\{T,f\}$ is a w.x_0-p.c. pair. It may be noted that T and f are not weakly commuting since $d(Tfx,fTx) > d(fx,Tx)$ for $\frac{1}{2} \leq x \leq 1$.

EXAMPLE 3. Let (X, d) be as in Example 2, $\emptyset(x)=x$, $T = I$ and

$$fx = \begin{cases} 5x, & x \leq 1 \\ 1 + x/2, & x > 1. \end{cases}$$

Since, at $y = 4$, $d(fy, y) = 1$ and $\emptyset(fy) - \emptyset(y) = -1$, the condition (P) of Corollary 3 does not hold for all $y \in X$. However, since

$$O(I,f;1/5) = \{fx_n : x_n = 1/5^{n+1},\ n = 1,2 \ldots\}$$

and $\quad d(fx_n, x_n) = 5^{-n} - 5^{-n-1} = \emptyset(fx_n) - \emptyset(x_n)$,

the condition (P) is satisfied for the orbit $O(I,f;1/5)$. So Corollary 3 applies to f, and $O(I,f;1/5)$ converges to 0, a fixed point of f. Indeed, the conclusions (2.1c) - (2.4c) are true for $x_0 = 1/5$. For $x_0 = 4$ we can have the following two orbits :

$$O_1 := \{4/5^n : n = 1, 2, 3 \ \ldots\}$$

and $\quad O_2 := \{2^n + 2 : n = 1, 2, 3, \ldots\}$.

It is easy to check that (P) is satisfied for O_1 and not for O_2. We land no where with O_2 while O_1 converges to 0, a fixed point of f.

EXAMPLE 4. This example shows that the conditions of Theorem 1 do not guarantee the existence of a common fixed point of maps T and f on X(=A). Let $X=\{x: x \geq 1/2\}$; $Tx = 2x$, $fx = 3x - 1/2$ and $\phi(x) = x$. Then $d(fx, Tx) = x-1/2 = \phi(fx) - \phi(Tx)$ for every x in X, and

$$O(T,f;1) = \{fx_n = 1 + (2/3)^{n-1} : x_n=1/2 + 2^{n-1}/3^n, \quad n=1,2,\ldots.\} \rightarrow 1.$$

Evidenty $T\frac{1}{2} = f\frac{1}{2} = 1$, and Theorem 1 applies to the maps of this example, and the coincidence point ½ is not a common fixed point of T and f. Indeed T and f are fixed point free maps.

EXAMPLE 5. This example shows that the conditions of Theorem 2 do not guarantee the existence of a common fixed point of T and f. Let $X = \{x : x \geq 1\}$, $Tx = x + 7/x$, $fx = (7x + 15)/8$, d the absolute value metric on X and

$$\phi(x) = \begin{cases} 22/x, & x = 1 \\ x, & x \neq 1. \end{cases}$$

Then $T(X) = [8, \infty) \subset [11/4, \infty) = f(X)$, and for $x_0 = 1$,

$$O(T,f;x_0) = \{ fx_n=8: x_n = 7, n = 1, 2, 3, ..\} \rightarrow 8.$$

Moreover, as $d(fx_n, Tx_n) = 0 = \phi(fx_n) - \phi(Tx_n)$, the (CP) of Theorem 2 holds for the orbit O(T,f;1). To check the weak 1-preorbital commutativity, we note that T and f are commuting at $x_n(=7)$, n = 1, 2, ... Thus Theorem 2 applies to the maps T and f, and 8 is a coincidence point of T and f. Evidently T and f have no common fixed points. Indeed, T is a fixed point free map. Note that T and f are not weakly commuting.

Recall that, as shown by Examples 4-5, the conditions of Theorems 1-2 are not sufficient to ensure the existence of a common fixed point of maps T, f on A = X. So we have the following :

QUESTIONS 1. Under what additional condition(s), maps T, f on A=X satisfying the conditions of Corollary 1/Theorem 1/Theorem 2 will have a common fixed point ?

2. Will corollary 2 yeild a common fixed point when T and f are continuous or commuting (Possibly both) ?

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FLOW OF BLOOD THROUGH A LOCALLY CONSTRICTED TUBE - A MATHEMATICAL MODEL AND ITS STATISTICAL ANALYSIS

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ABSTRACT

A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out mean to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosis. Therefore in this paper an attempt has been made to study the flow behaviour of the particulate suspension through a locally constricted tube and to obtain the wall shear stress distribution for different values of time, axial positions and number densities. A statistical analysis has been carried out in order to achieve the uncovering of the embedded relations between the various parameters in the study with the help of multiple regression and partial correlations.

Key words : Stenosis, Red Blood Cells, Shear Stress
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1. INTRODUCTION :

Interest in problem of mechanics of system with more than one phase has developed repidly in recent years. Frequently we are concerned with the motion of a liquid or gas, which contains a distribution of solid spherical particles. Such situation occur, for example, in the movement of dust-laden air, in problem offluidization, in environmental pollution in the process by which rain drops are formed by the coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence, combustion and more recently in blood flow studies.

Saffman [13] described the effect of dust by two parameters, namely the concerntration and relaxation time. These parameters measure the rate at which the velocity of a dust particle adjusts to change the fluid velocity and depends upon the size of individual particles. Michael and Miller [9], Liu [8] have studied the flow produced by the motion of an infinite plate in a dusty gas occupying semi-infinite spce above it. Rao [12], Nath [10], Verma and Mathur [14] have studied the flow of dusty viscous liquid in circular channels. Healy and Young [3] have obtained exact solution for the problem using Laplace transform technique. Gupta and Gupta [2] studied the flow of dusty gases through a channel with arbitrary time varying pressure gradient. Charya [1] studied the flow of dusty fluid through a constricted channel.

Recently, the attention of the researchers has also focussed on the study of flow behaviour of the suspension of particles through a locally constrticted tube in reference to an important disease termed as "stenosis" by medical scientists. Blood flowing through a stenotic vein develops greater resistance and drag on the tube walls and at times it happens to be fatal. Saffman's [13] dusty fluid serves as a better model to describe the blood as a binary system. Nayfeh [11] studied the pulsatile flow of liquid containing small solid particles through a long narrow circular tube. A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out means to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosios. Therefore, in this paper an attempt has been made to study the flow behaviour of the particulate suspensions through a locally constricted tube and obtained wall shear stress distribution, through the study under a mathematical model. The model has been used to construct synthetic data, on an electronic computer using the numerical inversion formula of Laplace transform. This data has been statistically studied to bring forward the implications of the model. More importantly, the statistical validation of the model could be done only with a reliable real-life data, which the authors very much missed to have had.

2. FORMULATION OF THE MATHEMATICAL MODEL :

In a circular tube, we consider the origin on the axis of the tube and the x-axis along the axis of the tube. The motion the dusty viscous, incompressible fluid is considered along the axis of the tube under the influence of the time (t) - dependent pressure gradient. The boundary conditions for the fluid are same as those for classical fluid and no particle boundary conditions are required. For the present, the initial conditions are at $t \leq 0$.

$$u = v = \frac{\partial u}{\partial t} = 0$$

The appropriate momentum equations in cylindrical polar coordinates are :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{KNo}{\rho}(v-u) \qquad \ldots (1)$$

$$m\frac{\partial v}{\partial t} = K(u-v) \qquad \ldots (2)$$

Where r is radial distance, u and v are velocities of fluid and dust particles respectively, K is the Stoke's resistance coefficient which, for spherical particles of radius r is $6\pi\mu r$. Also, ρ and u are the density and viscosity of the blood respectively, and No is the number density of red blood cells, p is the pressure. Radius of constricted region may be expressed as [11]:

$$R = a - \alpha_o\left(1 + \cos\frac{\pi z}{z_o}\right) \qquad \ldots (3)$$

where 'a' is the radius of non-constricted region, α_o is constant depending on lumen of tube, z_o is the axial distance and $2z_o$ is constricted length.

Introducing the following nondimensional quantities

$$r^* = \frac{r}{a},\; x^* = \frac{x}{a},\; p^* = \frac{pa^2}{\rho\nu^2},\; t^* = \frac{t\nu}{a^2},\; u^* = \frac{ua}{\nu},\; v^* = \frac{va}{\nu} \qquad \ldots (4)$$

In the equation (1) and (2) (droping the stars) we get

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \left(- \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) + B(v-u) \qquad \ldots (5)$$

$$\frac{\partial v}{\partial t} = \frac{u-v}{t} \qquad \ldots (6)$$

where $B = \frac{M}{\tau} = \frac{N_o K a^2}{\mu}$, $M = \frac{No\, m}{}$

$\tau = \frac{v\ m}{k a^2}$;

m being the mass of red blood cells, the radius

$$\beta = 1 - \alpha \left(1 + \cos \frac{\pi z}{z_o} \right) \qquad \ldots (7)$$

where, $\beta = \frac{R}{a}$, $\alpha = \frac{\alpha_o}{a}$

and the pressure gradient may be assumed in the form

$$\frac{\partial p}{\partial x} = C_1 F(t) = e^{-\lambda t} \text{ (say)} \qquad \ldots (8)$$

where, C_1 is a constant.

Equation (6) may be solved for v, using the particle initial condition $v(r,0) = 0$, to give

$$v(r,t) = \frac{1}{\tau} e^{-t/\tau} u(r,\eta) e^{\eta/\tau} d \qquad \ldots (9)$$

On substituting (8) and (9) in equation (5) we get

$$\frac{\partial u}{\partial t} = C_1 F(t) + \left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) + B\left[\frac{1}{\tau} e^{t/\tau} \int_0^{\tau} u(r,\eta) e^{\eta/\tau} d\eta - u \right] \qquad \ldots (10)$$

Once the above fluid momentum, equation is solved for u with the help of equation (3) for radius varying with axial distance, v can be obtained from equation (9).

3. SOLUTION OF THE PROBLEM :

Applying the Laplace transform in equation (5), we get

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \left(\frac{s^2 - BK_2}{s}\right)\bar{u} + c_1\bar{F}(s) = 0 \qquad \ldots (11)$$

where $K_2 = \frac{1}{\tau}$

Solving the equation (11) we get complimentary function as follows.

$$\bar{u} = aJ_0(Qr) + BY_0(Qr) \qquad \ldots (12)$$

where $Q = \left(\frac{s^2 + Bs - BK_2}{s}\right)^{1/2}$

J_0 and Y_0 are modified Bessel's functions.

Variation of the parameters method has been used to evaluate second order differential equation. We have obtained the solution for $\bar{u}$, using equation (12) as assumed solutions.

$\bar{u} = AJ_0\ (Qr) + BY_0\ (Qr)$

where

$$A = \frac{CY_{\cup}r^2}{Q(Y_0J_0' - J_0Y_0')}\,dr + a \qquad \ldots (13)$$

$$B = \frac{CJ_{\cup}r^2}{Q(Y_0J_0' - J_0Y_0')}\,dr + b \qquad \ldots (14)$$

As limit $r \longrightarrow 0$, $\bar{u}\ (r,s)$ should be finite, and at $r = \beta$ no slip condition should be satisfied. This leads to:

$$\bar{u} = D_1 J_0 (Qr) Y_0(Qr) - u_0 \qquad \dots (15)$$

where

$$D_1 = \frac{C}{Q^3} (r^2 - \frac{2}{Q^2}) + \frac{2C}{Q^5} \log (r^2 - \frac{2}{Q^2}) + a$$

Now the value of u (as limit $r \longrightarrow o, u(r,s)$ should be finite and at $r \longrightarrow \beta$ no slip condition should be satisfied), can be evaluated using Laplace inversion formula, as follows:

$$u(r,t) = \frac{1}{2\pi i} \int_{r-\infty}^{r+\infty} \frac{1}{(s+)} (D_1 J_0 (Qr) - Y_0 (Q_r) - \bar{u}_0) e^{st} ds \qquad \dots (16)$$

Therefore,

$$u(r,t) = -\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} [\frac{s^{5/2}}{(s^2 + Bs - BK_2)^{5/2}} (r^2 (s^2 + Bs - BK_2) - 2s)$$

$$+ 2 \log \frac{(r^2 (s^2 + Bs - BK_2) - 2s)}{(s^2 + Bs - BK_2)} + \frac{\bar{u}_0}{s} (1 + \frac{1}{2} \log (B, \beta^2)$$

$$+ \frac{1}{2} (\frac{s}{B} - \frac{K_2}{s}) - \frac{1}{4} (\frac{s}{B} - \frac{K_2}{s})^2 (\frac{1}{s+})$$

$$\{1 - \frac{1}{2} \log (B.r^2) - \frac{1}{2} (\frac{s}{B} - \frac{K_2}{s}) + \frac{1}{4} (\frac{s}{B} - \frac{K_2}{s})^2 \}] e^{st} ds$$

Then u(r,t) has been solved numerically by ussing inversion formula of Laplace transformation with the help of computer program [6]. Similarly v(r,t) can be found from equation (9) along with u(r,t). The skin friction on unit area of the surface of cylinder due to fluid is given by

$$F = \mu \left[\frac{\partial u}{\partial r} \right] r = 1 \qquad \ldots (17)$$

Once the value of v is obtained, the skin friction is also found out by equation (17), by putting the value of v.

Velocities of liquid and solid particles and the skin friction on the wall have been represented grahically with variable axial positions, time and number densities of the suspension.

From figure (1) it is clear that the velocity of the fluid decreases as the number density of the dust particles increases. Velocity of the fluid also decreases with the axial position from the throat and the time that elapses. Fluid velocity is always maximum on the central axix.

From figure (2) it is clear that the particle velocity decreases as the number density increases. Particle velocity also decreases with axial position from the throat and the time elapsed. Further, from figure (1) and (2), we infer that at a particular axial position the fluid velocity is greater than the particle velocity. Near the boundary, the dust particles exhibit sudden changing velocity and this tendency decreases with the number density.

From figure (3) we find that skin friction increases with number density and decrease with time and axial position from the throat. Thus, in order to have lower values of the skin friction for blood type suspension the number density should be decreased.

This is frequently done by the dilution process in medical sciences. The above inferences from the mathematical model are characterized in a much more comprehensive manner by the statistical analyses in the following section.

4. STATISTICAL ANALYSES :

For the data generated per the mathematical model as displayed in figures 1, 2 and 3, simple statistical analyses were carried out. Multiple regression of u, v and F each on the other relevent independent variables and various partial correlation coefficient ([4],[5], & [7]) were computed in order to bring out the dependence structures

First we take to 'u', i.e., the velocity of the fluid.

When $N = 4 \times 10^7$ ·

(a) Case 1 : z = 0 : It is found tnat,

$$u = 5.113 - 0.129t - 5.208r. \qquad \ldots (18)$$

Also, the partial correlations,

$$R_{rt}.u = -0.051 \; ; \; R_{ut}.r = -0.994 \; ; \; R_{ur}.t = -0.052 \ldots (19)$$

(b) Case 2 : z = 2 : we find that,

$$u = 2.044 - 0.252t - 2.084r. \qquad \ldots (20)$$

And,

$$R_{rt}.u = -0.216 \; ; \; R_{ut}.r = -0.992 \; ; \; R_{ur}.t = -0.218 \ldots (21)$$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$u = 5.861 - 0.752t - 5.237r. \qquad \ldots (22)$$

And

$$Rrt.u = -0.126 \,;\; Rut.r = -0.968 \,;\; Rur.t = -0.131 \qquad \ldots (23)$$

(b) Case 2 : $z = 2$: We find that,

$$u = 2.357 - 0.088t - 2.277r. \qquad \ldots (24)$$

And

$$Rrt.u = -0.088 \,;\; Rut.r = -0.995 \,;\; Rur.t = -0.088 \qquad \ldots (25)$$

So (18), (20), (23) and (25) each show the average mathematical way 'u' goes down with 't' and /or 'r' going up. Also, as per (19), (21), (24), &(25)'u' and 't' are significantly related (opposite way) while the other relations are insignificant.

Now we take to 'v', i.e., velocity of solid particles.

When, $N = 4 \times 10^7$:

(a) Case 1 : $z = 0$ we find that

$$v = 0.769 - 0.121t - 0.679r \qquad \ldots (26)$$

And

$$Rrt.v = -0.114 \,;\; Rvt.r = -0.944 \,;\; Rvr.t = -0.121 \qquad \ldots (27)$$

(b) Case 2 : $z = 2$: We find that,

$$v = 1.676 - 0.276t - 2.147r.$$

And,

$$Rrt.v = -0.022 \,;\; Rvt.r = -0.711 \,;\; Rvrr.t = -0.311 \qquad \ldots (28)$$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$v = 0.845 - 0.132t - 0.755r. \qquad \dots (29)$$

And,

$$R_{rt.v} = 0.143 \; ; \; R_{vt.r} = -0.963 \; ; \; R_{vr.t} = -0.149 \qquad \dots (30)$$

(b) Case 2 : $z = 2$: We find that,

$$v = 0.346 - 0.110t - 0.338r.$$

And,

$$R_{rt.v} = -0.643 \; ; \; R_{vt.r} = -0.996 \; ; \; R_{vr.t} = -0.646 \qquad \dots (31)$$

The structures of the relationship are quite similar to those 'u' except that the correlations between 'r' and 't' and 'r' and 'u' is relatively not that insignificant as in case of 'u'.

Now we take to 'F', i.e. the skin friction.

When, $N = 4 \times 10^7$: We find that,

$$F = 0.480 - 0.087t - 0.163z. \qquad \dots (32)$$

and,

$$R_{zt.F} = -0.370 \; ; \; R_{Ft.z} = -0.995 \; ; \; R_{Fz.t} = -0.372 \qquad \dots (33)$$

When, $N = 5 \times 10^7$: We found that,

$$F = 0.688 - 0.095t - 0.205z. \qquad \dots (34)$$

And,

$$R_{zt.F} = -0.187 \; ; \; R_{Ft.z} = -0.985 \; ; \; R_{Fz.t} = -0.189 \qquad \dots (35)$$

Here also like for the case of 'u' only 'F' and 't' are significantly related the other pairs, namely, 'z' and 't' and 'F' and 'z' are not so significantly related.

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LATTICE SUM OF ELECTRIC FIELD GRADIENTS IN ORTHORHOMBIC CRYSTAL

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ABSTRACT

Electric field gradient (FEG) is calculated at a nuclear site in the Simple Orthorhombic crystal. The method uses Euler Maclaurin (EM) Summation formula and makes Plane Wise summation in direct crystal space.

Key words : Electric field gradient, Lattice Sum Convergence, Simple Orthorhombic Crystal, Euler-Maclaurin formula, Double Series.

1. INTRODUCTION

In a previous paper (Verma et al. 1983) a new technique to sum the electric field gradient (EFG) over lattice points at a nuclear site in simple tetragonal crystal was introduced. The technique involves repeated use of Euler-Maclaurin (EM) formula to evaluate the infinite series and the same formula was used to estimate error terms from a related finite sum. The lattice sum is evaluated inthe direct crystal space keeping the numerical computations reasonably low, unlike most of the previous methods using Fourier transform in reciprocal space.

In the present paper we use the above method (Verma et al. 1983) to calculate the Electric Field Gradient (EFG) at the nuclear site in simple orthorhombic crystal.

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2. DETAILS OF CALCULATION

The lattice part of electric field gradient (EFG) eq_{latt} is defined as

$$eq_{latt} = \sum_{\lambda} \frac{ze}{4\pi\epsilon_o} \frac{3z_\lambda^2 - r_\lambda^2}{r_\lambda^5} - \frac{ze}{4\pi\epsilon_o V}\int \frac{3z^2 - r^2}{r^5}\, d^3r \quad \ldots(1)$$

where,

z = charge on each ion

e = electronic charge

ϵ_o = permittivity of vacuum

z = z-co-ordinate of the ion at th lattice point

r = distance of lattice point from the origin situated at the nucleus of interest

V = volume of the unit cell

The integration refers to the contribution from an assumed uniform background of negative charges. Both the terms in equation (1) are conditionally convergent, hence same boundary shape at infinity is to be assumed in summation and integration.

We employ plane wise summation which leads to a rapidly convergent expression of the sum. This means the crystal is assumed to be slab shaped with faces perpendicular to z-direction. With this boundry shape the integration $\int \frac{3z^2-r^2}{r^5}\, d^3r$ in equation (1) for the assumed boundary shape comes out to be $(-8\pi/3)$ (de Wette 1961).

In an orthorhombic lattice the lattice points are written as $(n_1a,\ n_2b,\ n_3c)$, where n_1, n_2, n_3 run through all integers.

The summation in equation (1) can be written as

$$S = \frac{ze}{4\pi\epsilon_o a^3} \sum_{n_1, n_2, n_3 = -\infty}^{\infty} f(n_1, n_2 n_3) \quad \ldots (2)$$

where,

$$f(n_1, n_2, n_3) = \frac{2n_3B^2 - n_1^2 - n_2^2 A^2}{(n_1^2 + n_2^2A^2 + n_3^2 B^2)^{5/2}},$$

$A = b/a$ and $B = c/a$

The summation over n_1 and n_2 are to be carried out before n_3 summation.

2.1 Contribution from base plane $(z = n_3c = 0)$

The Contribution to $[\frac{4\pi\epsilon_o a^3}{ze} S]$ from the base plane is

$$U_o = \sum_{n_1, n_2 = -\infty}^{\infty} g(n_1, n_2)$$

$$= -4 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} g(n_1, n_2) + 2 \sum_{n_1=1} g(n_1, 0)$$

$$- \frac{2}{A^3} \sum_{n_2=1}^{\infty} g(0, n_2) \qquad \ldots (3)$$

where,

$$g(n_1, n_2) = \frac{1}{(n_1^2 + n_2^2 A^2)^{3/2}}$$

The Euler-Maclaurin (EM) Summation formula (Hildebrand 1956) is

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)dx + \frac{1}{2} \{f(N) + f(M) + \int_M^N P_1(x) \frac{df(x)}{dx} dx \qquad \ldots (4)$$

where $P_1(x) = x - 1/2$ for $0 < X \leq 1$ and is defined outside this region by $P_1(x+1) = P_1(x)$.

Applying the EM formula (4) to (3) we get,

$$U_o = \frac{4}{A} \sum_{n_1=1}^{\infty} \frac{1}{n_1^2} \quad \frac{2}{A^3} \sum_{n_1=1}^{\infty} \frac{1}{n_1^3} - 4 \int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R \, dx$$

in which $R = \frac{d}{dx} g(n_1, x)$

Thus $U_o = -\frac{4}{A}\ \xi(2) - \frac{2}{A^3}\xi(3) + \epsilon_o(\infty)$ ... (5)

Where $\epsilon_o(\infty)$ contains all integrals involving $P_1(x)$ terms and $\xi(2) = \pi^2/6$, $\xi(3) = 1.20205690$ are the Riemann Zeta functions at 2 and 3 respectively.

To Estimate $\epsilon_o(\infty)$ we apply the above procedure to the finite sum

$$U_o(N) = -\sum_{n_1, n_2=-N}^{N} g(n_1, n_2)$$

$$= -4\sum_{n_1=1}^{N}\ \sum_{n_2=0}^{N} g(n_1, n_2) + 2\sum_{n_1=1}^{N} g(n_1, 0)$$

$$-\frac{2}{A^3}\sum_{n_2=1}^{N} g(0, n_2)$$

and get,

$$U_o(N) = -4\sum_{n_1=1}^{N} G(n_1, N) - \frac{2}{A^3}\sum_{n_2=1}^{N} g(0, n_2)$$

$$-2\sum_{n_1=1}^{N} g(n_1, N) - 4\int_o^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

Hence,

$$\epsilon_o(N) = -4\int_o^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

$$= -4\sum_{n_1=1}^{N}\ \sum_{n_2=0}^{N} g(n_1, n_2) + 4\sum_{n_1=1}^{N} G(n_1, N)$$

$$+ 2\sum_{n_1=1}^{N} g(n_1, 0) + 2\sum_{n_1=1}^{N} g(n_1, N) \quad ....(6)$$

where,

$$G(n_1\ ,\ N) \;=\; \frac{N}{n_1^{\,2}\ (n_1^{\,2}\ +\ N^2\ A^2\ )^{1/2}}$$

As N iucreases $|\ \epsilon_o\ (\infty)\ -\ \epsilon_o\ (N)\ |$ decreases. It was shown by Verma et al. (1993) that the absolute difference between ( ) and (N) for a similar expression reduces below 10 as N is taken greater than 80. Our expression has a similar nature and hence we expect that to the accuracy of present calculations (five significant digits) $|\ \epsilon_o\ (\infty)\ -\ \epsilon_o\ (80)\ |$ may be neglected. Hence we calculate (N) for N = 80 from equation (6) and substitute it for (oo) in equation (5) to get the contribution due to base plane.

2.2 Contribution from the plane $z = n_3c\ (n \neq 0)$

The contribution of the plane $Z=n_3c$ to $(4\pi\epsilon a_o^3 S/Ze)$ is

$$U_k = \sum_{n_1,n_2=-\infty}^{\infty} f(n_1,n_2,n_3)$$

$$= 4\sum_{n_1=1}^{\infty}\ \sum_{n_2=0}^{\infty} f(n_1,n_2,n_3)\ -\ 2\sum_{n_1=1}^{\infty} f(n_1,0,n_3)$$

$$+\ 2\sum_{n_2=0}^{\infty} f(0,n_2,n_3)\ -\ \frac{2}{K^3} \qquad \ldots\ (7)$$

where $K = n_3B$

Using EM formula (4) on summation over n_2 we get,

$$U_k = \frac{4}{A}\sum_{n_1=1}^{\infty}\frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A}\sum_{n_1=1}^{\infty}\frac{1}{(K^2+n_1^2)} + \frac{2}{AK^2}$$

$$+\ 2\int_0^{\infty} P_1(x)R_2dx\ +\ 4\int_0^{\infty} P_1(x)\ \sum_{n_1=1}^{\infty}\ R_1dx$$

$$= \frac{4}{A} \sum_{n_1=0}^{\infty} \frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A} \sum_{n_1=0}^{\infty} \frac{1}{(K^2+n_1^2)} - \frac{2}{AK^2}$$

$$+ 2\int_0^{\infty} P_1(x)\, R_2 dx + 4\int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R_1 dx \qquad \ldots (8)$$

where $R_1 = \frac{d}{dx} f(n_1, x, n_3)$ and $R_2 = \frac{d}{dx} f(0, x, n_3)$

The first two series of equation (8) can be summed exactly using Cauchy's Residue formula (Spiegel 1964).

$$\sum_{n=0}^{\infty} \frac{1}{(K^2 + n^2)^2} = \frac{\pi}{4} \left[\frac{\coth(\pi K)}{K^3} \quad \frac{\operatorname{cosech}^2(\pi K)}{K^2} \right] + \frac{1}{2K^4}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{K^2+n^2} = \frac{1+\pi K \coth(\pi K)}{2K^2}$$

So that equation (8) becomes

$$U_K = \frac{2\pi^2}{A} \operatorname{cosech}^2(\pi K) + e_k(\infty) \qquad \ldots (9)$$

where $e_k(\infty)$ contains all integrals involving $P_1(x)$ terms.

Again to estimate $e_k(\infty)$ we repeat the above procedure with the finite sum

$$U_k(N) = \sum_{n_1, n_2=-N}^{N} f(n_1, n_2, n_3)$$

$$= 4 \sum_{n_1=1}^{N} \sum_{n_2=0}^{N} f(n_1, n_2, n_3) - 2 \sum_{n_1=1}^{N} f(n_1, 0, n_3)$$

$$+ \sum_{n_2=0}^{N} f(0, n_2, n_3) - \frac{2}{K^3}$$

and get

$$e_K(N) = 2\int_0^{N} P_1(x)\, R_2 dx + 4\int_0^{N} P(x) \sum_{n_1=1}^{N} R_1 dx$$

$$= \sum_{n_1, n_2=-N}^{N} f(n_1, n_2, n_3) + H(n_1, n_3, N) \qquad \ldots (10)$$

where,

$$H(n_1, n_3, N) = - \sum_{n_1=1}^{N} \frac{12\,K^2 N}{(K^2+n_1^2)^2 (K^2+n_1^2+N^2A^2)^{1/2}}$$

$$+ \sum_{n_1=1}^{N} \frac{4K^2\,N^3\,A^2}{(K^2+n_1^2)\,(K^2+n_1^2+N^2A^2)^{3/2}}$$

$$+ \sum_{n_1=1}^{N} \frac{4\,N}{(K^2+n_1^2)\,(K^2+n_1^2A^2)^{\frac{1}{2}}}$$

$$- 2 \sum_{n_1=1}^{N} f(n_1, n_3, N) - f(0, n_3.N)$$

$$- \frac{4\,N}{K^2(K^2+N^2A^2)^{\frac{1}{2}}} \quad \frac{2\,N^3\,A^2}{K^2(K^2+N^2A^2)^{3/2}}$$

We have set N=80, and evaluated $e_k(N)$ from equation (10). This is substituted for $\epsilon_k(\infty)$ in equation (9) thereby neglecting $|\,\epsilon(\infty) - e_k(N)\,|$ and obtaining U_k.

3. Results and discussion

We made calculations for eight orthorhombic structures with different ratios of a:b:c (listed in table-1). Sets of lattice parameters correspond to the lattices of orthorhombic elements B, Ga, P(black), O, S(yellow), Br, I and U (Darken Lawrence et al.1953).

The contributions from base plane and other planes were evaluated from the equations derived above. The contribution falls rapidly as n_3 increases and that is the speciality of this method. In all our claculations a maximum of six planes above the origin need to be considered. The contribution from the planes below the origin was equal to that from the planes above by symmetry. The contribution $(-8\pi/3 \ \frac{ze}{4\pi\epsilon_o a^3})$ was added to get the eq_{latt} in c-direction.

For each set of lattice parameters the calculations were repeated thrice by choosing the z axis along each of the three edges of the cell and hence to get the EFG's along the three direction. These directions were then labelled according to the convention $|V_{zz}| \geq |V_{yy}| \geq |V_{xx}|$. The sum $V_{xx} + V_{yy} + V_{zz}$ is found to be zero within the limits of accuracy of present calculations as is expected from the Laplace equation.

As the total computations involved is of low order and the sum is made in direct crystal space without any special regrouping of charges, the use of Euler Maclaurin summation formula difinitely has interesting application in lattice summation exercises.

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TABLE - 1

Lattice EFG in Orthorhombic Lattice Structures at Substitutional Sites in units of Ze/4 C_o

Element	Lattice Parameters of (in K_x units) the simple orthorhombic Lattice			V_{xx}	V_{yy}	EFG V_{zz} = eq
	a	b	c			
B	8.93	10.13	17.86	0.0013986	0.0039938	0.0053920
Ga	4.5107	4.516	7.6448	0.022063	0.022318	-0.044399
P (black)	3.31	4.38	10.50	0.010841	0.10415	-0.11499
O	3.44	3.82	5.50	0.017592	0.056615	-0.074201
S (yellow)	10.48	12.92	24.55	0.00047180	0.0028624	-0.0033361
Br	6.67	8.72	4.48	-0.0098774	-0.027391	0.037269
I	7.2501	9.7711	4.7761	-0.0082761	-0.023389	0.031667
U	5.865	4.945	2.852	-0.061142	-0.095374	0.15652

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EULER MACLAURIN SUMMATION FORMULA FOR TWO DIMENSIONS

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ABSTRACT

We here develop Euler Maclaurin Summation Formula (EMSF) for two dimension case so that it can be used in summing over a set of lattice points in a plane. Other three versions of the formula are also got.

AMS(MOS) Subject classifications (1980): 65 B 15

Key words : Lattice sum, Discrete mathematics, Double series summation.

1. INTRODUCTION

The EMSF for one dimension [1] is

$$\sum_{r=1}^{n} f(r) = \int_{1}^{n} f(x)dx + \frac{1}{2}\{f(1) + f(n)\} + \int_{1}^{n} P_1(x)\, f'(x)\, dx$$

where $P_1(x)$ is Bernoulli's polynomial of order one. But no such result like Euler Maclaurin Summation Formula (EMSF) is available in problems of two and three dimensions. We here develop the formula for two dimensions and similar extension for three dimensions is easy.

2. FORMULATION OF THE PROBLEM

Here we find the value of

$$S = \sum_{r=1}^{m} \sum_{s=1}^{n} f(r,s)$$

we first fix r and let s vary from 1 to n, thus

$$S = \sum_{r=1}^{m} \int_{1}^{n} f(r,y)dy + \frac{1}{2}\sum_{r=1}^{m} f(r,1) + \frac{1}{2}\sum_{r=1}^{m} f(r,n)$$

$$+ \sum_{r=1}^{m} \int_{1}^{n} P_1(y) f_y(r,y)dy$$

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we apply EMSF to each term of the R.H.S. and after arranging, get

$$S = \int_1^m \int_1^n f(x,y,)dxdy + \frac{1}{4}\{f(1,1)+f(m,1)+f(1,n)+f(m,n)\}$$

$$+\frac{1}{2}\{\int_1^n f(1,y)dy + \int_1^n f(m,y)dy + \int_1^m + f(x,1)dx + \int_1^m f(x,n)dx\}$$

$$+\frac{1}{2}[\int_1^n P_1(y)f_y(1,y)dy + \int_1^m P_1(x)f_x(x,1)dx +$$

$$+\int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,1)dx]$$

$$+\int_1^m \int_1^n \{P_1(x)f_x(x,y) + P_1(y)f_y(x,y)\}dxdy$$

$$+\int_1^m \int_1^n P_1(x)P_1(y)f_{xy}(x,y)dxdy \qquad \ldots (1)$$

provided the partial derivatives used exist and are continuous.

In Green's theorem

$$\iint_D (Q_x - P_y)dxdy = \int_C (P\,dx + Q\,dy)$$

let C be the rectangle with vertices (1,1), (m,1), (m,n), (1,n) let D be the part of the plane enclosed by C and let $Q = -f(x,y) = -P$. then

$$\iint_D \{f_x(x,y) + f_y(x,y)\}\,dxdy = \int_C f(x,y)\,(dy-dx)$$

$$= -\int_1^m f(x,1)dx + \int_1^n f(m,y)dy + \int_1^m f(x,n)dx - \int_1^n f(1,y)dy \qquad \ldots (2)$$

Next we put $P = -P_1(x)f_x(x,y)$ and $Q = P_1(y)f_y(x,y)$, then

$$\int_1^m \int_1^n \{P_1(y)f_{xy}(x,y)+P_1(x)f_{xy}(x,y)\}dxdy$$

$$= -\int_1^m P_1(x)f_x(x,1)dx + \int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,n)$$

$$- \int_1^n P_1(y)f_y\,(1,y)dy. \qquad \ldots (3)$$

Thus (1) becomes, after putting values from (2) and (3)

$$S = \int_1^m \int_1^n \{f(x,y) - \frac{1}{2}\,f_x(x,y) - \frac{1}{2}\,f_y(x,y)\}dxdy$$

$$+ \frac{1}{4}\,\{f\,(1,1) + f\,(m,1) + f(1,n) + f\,(m,n)\}$$

$$+ \int_1^m \{f\,(x,n)\;dx + \int_1^n f(m,y)\;dy$$

$$+ \int_1^m P_1(x)\;f_x(x,n)\;dx + \int_1^n P_1(y)f_y(m,y)\;dy$$

$$+ \int_1^m \int_1^n \{P_1(x)f_x\;(x,y) + P_1(y)f_y\;(x,y)\}\;dxdy$$

$$+ \int_1^m \int_1^n [P_1(x)\;P_1(y) - \frac{1}{2}\;\;P_1(x)-\frac{1}{2}P_1(y)]f_{xy}(x,y)dx\,dy$$

$$\ldots (4)$$

This is our formula in most general case. We now try to discuss EMSF in two dimension for skew-symmetric and symmetric functions. We know that every function can be expressed as a sum of a symmetric and skew-symmetric function. Hence let

$$f(x,y) = g\,(x,y) + h(x,y),$$

g(x,y) being symmetric and h(x,y) skew symmetric.

The terms of

$$\sum_{r,s=1}^{n} h(r,s)$$

can be grouped in pairs of type h(r,s) + h(s,r) and we are left with terms of the type h(r,r), each of which is zero by skew symmetric property. Thus

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

Now g is symmetric i.e. g(x,y) = g(y,x) Differentiating,

$g_x(x,y) = g_x\,(y,x)$. Using these results and putting m=n, (1) reduces to

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n\int_1^n g(x,y)\,dxdy + \frac{1}{4}\,\{g(1,1) + 2g(n,1)$$

$$+ g(n,n)\} + \{ \int_1^n g(x,1)\,dx + \int_1^n g(x,n)\,dx\}$$

$$+ \{\int_1^n P_1(x)\, g_x(x,1)\,dx$$

$$+ \int_1^n P_1(x)\, g_x(x,n)\,dx\}$$

$$+ 2 \int_1^n \int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$+ \int_1^n \int_1^n P_1(x)\, P_1(y)\, g_{xy}(x,y)\,dxdy \qquad (5)$$

The last result can be written in terms of 3rd order Bernoulli polynomials if we integrate by parts twice the terms containing $P_1(x)$, and note that $P_3(x)$ is zero at integral arguments. So

$$\int_1^n P_1(x)\, \{g_x(x,1) + g_x(x,n)\}\,dx$$

$$= \frac{B_2}{2!} \{g_x(n,n) - g_x(1,1)\} + \int_1^n P_3(x)\, [g_{xxx}(x,1)$$

$$+ g_{xxx}(x,n)]\,dx \qquad \ldots (6)$$

$$\int_1^n \int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$= \frac{B_2}{2!} \int_1^n \{g_x(n,y) - g_x(1,y)\}\,dy$$

$$+ \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)\, dxdy \qquad \ldots (7)$$

$$\int_1^n \int_1^n P_1(x)\, P_1(y)\, g_{xy}(x,y)\, dxdy$$

$$= \frac{B_2}{2\,!} \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}\ (1,1)\}$$

$$+ B_2 \int_1^n P_3(x)\ \{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\ dx$$

$$+ \int_1^n \int_1^n P_3(x)\ P_3(y)\ g_{xxxyyy}(x,y)\ dxdy \qquad \ldots (8)$$

provided the partial derivatives used exist and are continuous so that change of order of differentiation is permitted.

Putting in (5) the above values as obtained in (6), (7) and (8) and arranging, we have,

$$\sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n \int_1^n g(x,y)dxdy + \frac{1}{4}\ \{g(1,1) + 2g(n,1) + g(n,n)\}$$

$$+ [\ \frac{B_2}{2!}\ \{g_x(n,n) - g_x(1,1)\}$$

$$(\frac{B_2}{2!})^2\ \ \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}(1.1)\}]$$

$$+ \int_1^n \{g(x,1) + g(x,n)\ \ dx$$

$$+ B_2 \int_1^n \{g_x(n,y) - g_x(1,y)\}\ dy$$

$$+ \int_1^n P_3(x)\{g_{xxx}(x,1) + g_{xxx}(x,n)\}\,dx$$

$$+ B_2 \int_1^n P_3(x)\,\{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\,dx$$

$$+ 2 \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)\,dxdy$$

$$+ \int_1^n \int_1^n P_3(x) P_3(y) g_{xxxyyy}(x,y)\,dxdy \qquad \ldots (9)$$

3. APPLICATION

A number of physical quantities are expressed as a negative powers of distance. Matter is a collection of discrete atoms. In rough calculation, matter is taken to be continuous and the physical quantities are calculated using double or triple integrals. This is done because much dificult mathematics is required if we consider the atoms at the positions where they are actually situated. Some people attack such a situation by a number transformations. If one wants to avoid transformations and avoid the assumption of continuous matter then one way of handling the situation is through the theorem developed in this paper. Here we give one such application using fourth power of distance.

I. J. Zukker and M.M. Robertson [2] calculated

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 6.0267002 \pm .00207895$$

using a number of Transformations and L functions. We, in order to calculate the above sum, take

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 4\zeta(4)+4\sum_{m,n=1}^{\infty}\frac{1}{(m^2+n^2)^2}$$

$$= 4\zeta(4) + 4\sum_{m,n=1}^{5}\frac{1}{(m^2+n^2)^2}$$

$$+ 8\sum_{n=1}^{5}\sum_{m=6}^{\infty}\frac{1}{(m^2+n^2)^2}+4\sum_{m,n=6}^{\infty}\frac{1}{(m^2+n^2)^2}$$

The second term is a finite sum, the third has been calculated using EMSF for one dimension and the last one using our theorem for two dimensions. The result got is

6.0265284 ± 0.005736

which compares well with [2].

4. REMARKS

The formula has a wide potentiality of application as most physical quantities are expressible as a point function is two or three dimensions. In many cases, form (2) is more suitable for numerical calculations than form (1).

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ON SOME SELF-SUPERPOSABLE FLUID MOTIONS IN PROLATE SPHEROIDAL DUCTS

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ABSTRACT

In the present paper an attempt has been made to find some self-superposable motions of incompressible fluid in prolate spheroidal ducts. No stress has been given on boundary conditions and the solutions thus determined contain a set of constants. Pressure distribution of some of such flows has also been discussed. The curves along which the vorticity of the flow becomes constant have also been attempted. Some irrotational flows and the surfaces along which the flows show the tendency of irrotationality has also been determined. The aim of the paper is to introduce a method for solving the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self-superposability.

1. INTRODUCTION

It was shown by Ballabh [2] that when a flow with velocity q satisfies the condition

$$\text{Curl } (\bar{q} \times \text{curl } \bar{q}) = 0 \qquad \ldots (1)$$

it becomes self-superposable. In the present paper the authors have attempted some fluid velocities for which $(\bar{q} \times \text{Curl } \bar{q}) = \bar{p}$ (say) can be represented as the gradient of a scalar quantity. Attention has been focussed on incompressible fluids only. For such fluids some velocities have been determined which satisfy the condition in prolate spheroidal system of coordinates and

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for which $\bar{p}$ can be represented as the gradient of a scalar quantty Θ (say). The solutions thus found must be of self-superposable type. Each will have a set of constants which can be determined by the boundary conditions.

It has further been shown by Ballabh (3) that if $\bar{q}_1$ and $\bar{q}_2$ are self-superposable flows and $\bar{q}_1 \pm \bar{q}_2$ are also self-superposalbe then $\bar{q}_1$ and $\bar{q}_2$ are mutually superposable. By using this property some more self-superposable flows have been determined. Attempts have also been made to find some curves along which the vorticity of the flow becomes constant and also the conditions of irrotationality.

In this paper a method will be introduced to solve the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self superposability. Mittal [5,6] Mittal et al.[7,8,9,10] have recently solved the equations for different coordinate systems.

FORMULATION OF THE PROBLEM

Let $\bar{q}$ x Curl $\bar{q} = \bar{p}$ ...(2)

For incompressible fluids, we have

$$\text{div } \bar{q} = 0 \qquad \ldots (3)$$

Now if a solution of equation (3) be found in such a way that p can be represented as the gradient of a scalar quantity say Θ, i.e.,

$$\bar{p} \text{ grad } \Theta, \qquad \ldots (4)$$

it will give a self-superposable flow. It has been shown by Agarwal [1] that such solutions will also satisfy the equation of motion for a steady flow. For determining a flow of liquid let us consider the flow in prolate spheroidal coordinates (u, v, w). If q_u, q_v, q_w be the components of q at any point (u, v, w) in prolate spheroidal coordinates [11] then in order to make equation (3) integrable we may consider the following cases :

Case I

Let $q_u = 0$. In this case equation (3) will be satisfied by a solution.

$$
\begin{aligned}
q_u &= 0\\
q_v &= \frac{AU(u)\ W(w)}{\sinh u.\ \sin v\ \sqrt{(\sinh^2 u + \sin^2 v)}} \qquad \ldots (15)\\
q_w &= \frac{BU_1\ (u)\ V(v)}{(\sinh^2 u + \sin^2 v)}
\end{aligned}
$$

where $U(u)$, $U_1(u)$ are integrable functions of u, $V(v)$ and $W(w)$ the integrable functions of v and w respectively, and A, B the constants.

For this fluid velocity, it can be shown that $\bar{p}$ can be represented by the gradient of a scalar quantity θ given by

$$
\begin{aligned}
\theta = {} & \frac{A^2W^2}{a \sin^2 v} \int \frac{U\ U'\ du}{\sinh^2 u\ (\sinh^2 u + \sin^2 v)^{3/2}}\\
& - \frac{A^2\ W^2}{a \sin^2 v} \int \frac{U^2\ \text{cosech}\ u.\ \text{Coth}\ u}{(\sinh^2 u + \sin^2 v)^{3/2}}\ du\\
& + \frac{B^2 U_1^2}{a \sinh u} \int \frac{V\ V'\quad dv}{\sin v\ (\sinh^2\ u + \sin^2\ v)^2}\\
& + \frac{B^2\ U_1^2}{a \sinh u} \int \frac{V^2\ (\sinh^2\ u - \sin^2 v)\ \cos v}{(\sinh^2 u + \sin^2 v)^3\ \sin^2 v}\ dv
\end{aligned}
$$

$$+ \frac{A\,B\,U\,U_1}{a(\sinh^2 u+\sin^2 v)^4} \; [\{V' + V \operatorname{Cot} v\} (\sinh^2 u+\sin^2 v) + V \sin 2v] \int W \, dw$$

$$- \frac{A^2\,U^2}{a(\sinh^2 u+\sin^2 v)^2 \sinh^2 u \, \sin^2 v} \int W\,W' \, dw \qquad \ldots (6)$$

here U(u), $U_1(u)$, V(v), W(w) are represented by U, U_1, V and W respectively and U', U_1', V' and W' represent their differentials.

By choosing different suitable sets of values of U, U_1 V and W we may get a number of self-superposable fluid velocities. One of such velocities can be obtained by taking

$$U_1 = U = a \sinh u, \; V = a \sin v, \; W = a \cos w$$

$$\text{and } B = A \qquad \ldots\ldots (7)$$

the fluid velocity will become ;

$$q_u = 0$$

$$q_v = \frac{A \operatorname{Cos} w}{\sin v \sqrt{(\sinh^2 u+\sin^2 v)}} \qquad \ldots\ldots (8)$$

$$q_w = \frac{A \sinh u . \sin v}{(\sinh^2 u + \sin^2 v)}$$

and ;

$$\Theta = \frac{A^2}{a} \left[\frac{\operatorname{Cos}^2 w}{\sin^4 v} \left\{ \frac{1}{2 \sin v} \log \left(\frac{\sqrt{\sinh^2 +\sin^2 v} - \sin v}{\sqrt{\sinh^2 u+\sin^2 v}+\sin v} \right) - \frac{1+\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right\} + 2\sinh u \left\{ \frac{\sqrt{\sinh^2 u+\sin^2 v}+\sinh^2 u}{\sqrt{\sinh^2 u + \sin^2 v}} \right\}^{1/2} \right.$$

$$+ \frac{\sinh^2 u \, \mathrm{Cos}\, v \sin w(2 \sinh^2 u + 3 \sin^2 v)}{(\sinh^2 u + \sin^2 v)}$$

$$- \frac{. \sin^2 w}{2(\sinh^2 u + \sin^2 v)^2} \qquad \ldots (9)$$

If U, U_1, V, W are constants then

$$q_u = 0$$

$$q_v = \frac{C_1}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (10)$$

$$q_w = \frac{D_1}{(\sinh^2 u + \sin^2 v)}$$

and

$$\theta = \frac{D_1^2}{a \sin^6 v} \left[\frac{3 \sin^{-1}}{2 \sin v} \left(\frac{\sin v}{\sqrt{\sin^2 u + \sin^2 v}} \right) - \frac{\sinh u}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sinh u} \right]$$

$$- \frac{2D_1^2 \sinh u}{a \sin^3 v \sqrt{(\sinh^2 u + \sin^2 v)}} +$$

$$\frac{D_1^2}{a \sinh^6 u} \left[\frac{3}{2 \sinh u} \sin^{-1} \left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right) - \frac{\sin v}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sin v} \right]$$

$$- \frac{2 D_1^2 \sin v}{a \sinh^3 u \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$+ \frac{C_1 D_1}{a} \left[\frac{\sinh u . \operatorname{Cot} v \, (\sin^2 v - \sinh^2 u)}{(\sinh^2 u + \sin^2 v)^3} \right] w \qquad \dots (11)$$

Case II

(i) When $q_v = 0$, In this case, the self-superposable flows may be

$$q_u = \frac{A_1 V_1 W_1}{\sinh u \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0$$

$$q_v = \frac{B_1 U_2 V_2}{(\sinh^2 u + \sin^2 v)} \qquad \dots (12)$$

and

$$\theta = - \frac{A_1 B_1 V_1 V_2 W_1'}{a \sin^3 v} \int \frac{U_2 \, du}{\sinh^3 u \, (\sinh u + \sin^2 v)}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2^2 (\sin^2 v - \sinh^2 u) \operatorname{Cosh} u \, dU}{\sinh^2 u \, (\sinh^2 u + \sin^2 v)^3}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2' U_2 \, du}{\sinh u \, (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2^2 \operatorname{Cos} v \, (\sinh^2 - \sin^2 v)}{\sin^2 v \, (\sinh^2 u + \sin^2 v)^3} dv$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2 V_2' \, dv}{\sin v (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{A_1^2 W_1^2}{a \sinh^2 u} \int \frac{(V_1 V_1' \operatorname{Cosec} v - V_1^2 \operatorname{cosec} v \cot v)}{\sin v(\sinh^2 u + \sin^2 v)} dv$$

$$+ \frac{A_1^2 V_1^2}{a \sinh^2 u \sin^2 v (\sinh^2 u + \sin^2 v)^2} \int W_1 W_1' \, dw$$

$$- \frac{U'_2 \sinh u}{(\sinh^2 u + \sin^2 v)} \;] \int W_1 \, dw \qquad \ldots. (13)$$

(ii)

$$q_u = \frac{A_1 \operatorname{Cos} w}{\sinh u . \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$q_v = 0 \qquad \ldots (14)$$

$$q_w = \frac{A_1 \sinh u . \sin v}{(\sinh_u^2 + \sin^2 v)}$$

and

$$\theta = - \frac{A_1^2 \sin w}{a \sin^3 v} \; [\operatorname{Coth} u - \frac{2}{\sin^2 v} \tan^{-1} (\frac{\tanh u}{\tan v})]$$

$$+ \frac{A_1^2 \sin v}{a} [\frac{2 \sinh u}{(\sinh^2 u + \sin^2 v)} - \frac{(2 \sin^2 v + 1)}{\sin v}$$

$$\sin^{-1} (\frac{v}{\sqrt{\sinh^2 + \sin^2 v}}) - \frac{1}{4 \sin^3 v} \sin (\frac{2 \sin v}{\sqrt{\sinh^2 u + \sin^2 v}})$$

$$+ \frac{A_1^2}{a} \operatorname{Cosec} v - \log (\frac{\sinh^2 u}{\sqrt{\sinh^2 u + \sin^2 v}})$$

$$\mathrm{Sin}^{-1}\left(\frac{u}{\sqrt{\mathrm{Sinh}^2+\sin^2 v}}\right) - \frac{1}{4\sinh^3 u}\sin\left(\frac{2\sinh u}{\sqrt{\sinh^2 u \div \sin^2 v}}\right)$$

$$+ \frac{A_1^2}{2a}\ \mathrm{cosech}^2 u\left[\frac{\sinh u.\ \sin v}{(\sinh^2+\sin^2 v)} - \sin^{-1}\left(\frac{\sinh u}{\sqrt{\mathrm{Sinh}^2 u+\sin^2 v}}\right)\right]$$

$$- \frac{A_1^2}{2a}\ \frac{\sin^2 w}{\sinh^2 u(\sinh^2 u+\sin^2 v)^2}$$

$$- \frac{2A_1^2}{a}\ \frac{\sin^4 v\ \mathrm{Cosh}\ u\ \sin w}{(\sinh^2 h + \sin^2)^3} \qquad \ldots \quad (15)$$

(iii)

$$q_u = \frac{C_2}{\sinh u\ \sin v\ \ \sinh^2 u+\sin^2 v}$$

$$q_v = 0$$

$$q_w = \frac{2}{(\sinh^2 u+\sin^2 v)} \qquad \ldots \quad (16)$$

and

$$\theta = \frac{D_2^2}{a\sin v}\left[\mathrm{Cosech}^7 u\ \{\sin v.\ \mathrm{Cosech}\ u\right.$$

$$- \frac{1}{32}\sin\left(4\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)\right)$$

$$- \frac{\sin v.\sinh u}{(\sinh^2 u+\sin^2 v} - \frac{(9+8\sin^4 v)}{8}\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

$$\left. - \frac{\sinh u}{\sin^2 v\ (\sinh^2 u+\sin^2 v)}\right]$$

$$+ \frac{D_2^2}{a\sinh u}\left[\mathrm{Cosech}^7 u\ \{\sinh u.\ \mathrm{Cosec}\ v\right.$$

$$- \frac{1}{32} \sin \left(4 \sin^{-1} \left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}}\right)\right)$$

$$- \frac{\sin v.\ \sinh u}{(\sinh^2 u+\sin^2 v)} - \frac{(9+8 \sinh^2 u)}{8} \sin^{-1}\left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}}\right)$$

$$- \frac{\sin v}{\sinh^2 u\ (\sinh^2 u + \sin^2 v)} \ ]$$

$$+ \frac{C_2 D_2 \sin v.\coth u\ (\sin^2 v - \sinh^2 u)\, w}{(\sinh^2 u + \sin^2 v)} \qquad \text{.... (17)}$$

Case III :- When $q_w = 0$, some self-superposable flow may be:

$$\text{(i)} \quad q_u = \frac{A_2\ V_3\ W_2}{\sinh u.\sin v \sqrt{\sinh^2 u+\sin^2 v}}$$

$$q_v = \frac{B_2\ U_3\ W_3}{\sinh u.\sin v \sqrt{\sinh^2 i+\sin^2 v}}$$

$$q_w = 0 \qquad \text{... (18)}$$

and

$$\theta = \frac{B_2^2\ W_3^2}{a \sin^2 v} \int \frac{(U_3'\ U_3 \sinh u - U_3^2 \cosh u)\, du}{\sinh^3 u\ (\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_2}{a \sin^3 u} \{V_3' \sin v - V_3 \cos v\} \int \frac{U_3 \operatorname{Cosech}^2 u \quad du}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_2}{a \sin^3 v} \{V_3' \sin v - V_3 \operatorname{Cos} v\} \int \frac{U_3 \operatorname{Cosech}^2 u \quad du}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_3}{a \sin^3 v} \{U_3' \sinh u - U_3 \operatorname{Cosh} u\} \int \frac{V_3 \operatorname{Cosec}^2 v \quad dv}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$+ \frac{A_2^2 W_2^2}{a \sinh^2 u} \int \frac{(V_3' V_3 \sin v - V_3^2 \operatorname{Cos} v)}{\sin^3 v \, (\sinh^2 u + \sin^2 v)^{3/2}} \, dv$$

$$+ \frac{A_2^2 V_3^2}{a \sinh^2 \sin^2 v \, (\sinh^2 u + \sin^2 v)} \int W_2 W_2' \, dw$$

$$+ \frac{B_2^2 U_3^2}{a \sinh^2 u \sin^2 v \, (\sinh^2 u + \sin^2 v)} \int W_3 W_3' \, dw \qquad \ldots (19)$$

(ii)

$$q_u = \frac{A_2 \operatorname{Cos} w}{\sinh u \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = \frac{A_2 \operatorname{Cos} w}{\sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (20)$$

$$q_w = 0$$

and

$$\Theta = \frac{A_2^2}{2a} \operatorname{Cosech}^2 u . \operatorname{cosec}^2 v . \cos^2 w \qquad \ldots. (21)$$

(iii)

$$q_u = \frac{C_3}{\sinh u \quad \sinh^2 u + \sin^2 v)} \qquad \ldots (22)$$

$$q_v = \frac{D_3}{\sin v \, (\sinh^2 u + \sin^2 v)}$$

$$q_w = 0$$

$$\theta = \text{Constt.} \qquad \ldots (23)$$

Case IV

When $q_u = q_v = 0$ a possible solution of equation (3) is given by

$$q_u = 0$$

$$q_u = 0$$

$$q_w = \frac{A_3\ U_u\ V_u}{(\sinh^2 u + \sin^2 v)} \qquad \ldots (24)$$

Case V

When $q_v = 0$, $q_w = 0$, the self superposable flow is given by

$$q_u = \frac{A_4\ V_5\ W_4}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0 \qquad \ldots (25)$$

$$q_w = 0$$

Case VI

When $q_w = 0$, $q_u = 0$, the flow is

$$q_u = 0$$

$$q_v = \frac{A_5\ U_5\ U_5}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (26)$$

$$q_w = 0$$

In all the above cases U_n, V_n, W_n (n = 1,2,3,4 .....) are integrable functions of u, v and w respectively and A_n, B_n, C_n, D_n (n=1, 2, 3,....) are constants whch may be determined by boundary conditions.

SUPERPOSABLE FLUID MOTION

It has already been shown that the hydrodynamic flows given by equations (5) and (25) are self-superposable. It can also easily be shown that if, $\bar{q}_1$ and $\bar{q}_2$ be the two flows given by equation (5) and (25), then $\bar{p}$ for $\bar{q}_1 \pm \bar{q}_2$ can also be represented by the gradient of scalar quantities. Thus $\bar{q}_1$ and $\bar{q}_2$ will be mutually superposable and a flow

$$q_u = \frac{A\, V(v)\, W(w)}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \dots (27)$$

$$q_v = \frac{B\, U(u)\, W(w)}{\sinh u .\ \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_w = \frac{C\, U(u) .\ V(v)}{(\sinh^2 u + \sin^2 v)}$$

is possible. The same same flow can be determined by mutually superposing the flows (12) and (26) and (18) and (24).

PRESSURE DISTRIBUTION

It is interesting to note that 0 is nothing but Bernoulli function given by (4)

$$\theta = \left(\frac{q^2}{2\, g}\right) + h + \frac{p}{g} \qquad \dots (28)$$

where q, g, h and p denote velocity, acceleration due to gravity, height above some hoprizontal plane of reference and the pressure head.

It is a well known fact that for an incompressible fluid the pressure head p is given by [4]

$$p = \frac{P}{\rho_o} + \text{constant.}$$

where P is the pressure distribution.

Also if the motion of the fluid be steady and slow then the value of h can be taken, without much loss of generality as u for the flows (5), (8) and (10); as v for the flow (12), (14) and (16); and w for (18), (20) and (22).

Thus for the flow (20) the pressure distribution

$$= K_1 + K_2 w + K_3 \mathrm{Cos}^2 w \frac{(\mathrm{Cosech}^2 u + \mathrm{Cose}^2 v)}{(\sinh^2 u + \sin^2 v)}$$

$$+ K_4 \,\mathrm{Cosech}^2 u \,\mathrm{cosec}^2 v \cos^2 w \qquad \ldots (29)$$

Similarly for the flow (22) taking $C_3 = D_3$

we have

$$P = K_5 + K_6 w + K_7 \frac{(\mathrm{cosech}^2 u + \mathrm{cosec}^2 v)}{(\sinh^2 u + \sin^2 v)} \ldots (30)$$

where $K_1, K_2, K_3, K_4, K_5, K_6, K_7$ are constants.

Similarly the pressure distribution for the other flows can be determined.

VORTICITY OF THE FLOW

It was shown by Ballabh [3] that for a self-superposable flow, verticity is constant along its stream lines. If $\bar{T}$ is a unit tangent along a stream line then

$$T \times q = 0 \qquad \ldots (31)$$

By equation (16) and (31) it can readily be shown that

$$T = \left[\frac{\sinh^2 u + \sin^2 v}{\{\sin^2 v(\sinh^4 + \sinh^2 u) + \sinh^4 u + \sin^4 v\}^{1/2}}, 0, \frac{\sinh^2 . \sin v}{\sin^2 v(\sinh^4 u + \sinh^2 u) + \sinh^4 u + \sin^4 v\}^{1/2}} \right] \qquad \ldots (32)$$

Hence the vorticity of the flow (16) is constant along the curve represented by equation (32). Similarly the curves of constant velocity can also be found for other flows,

IRROTATIONALITY

Vorticity $\bar{G}$ for the flow (10) can be calculated

$$\bar{G} = \frac{D_1}{a \sin v(\sinh^2 u+\sin^2 v)^{1/2}} \left\{ \frac{(\sinh^2 u-\sin^2 u) \operatorname{Cos} v}{(\sinh^2 u + \sin^2 v)^2} \right\} \bar{e}_1$$

$$- \frac{D_1}{a \sinh u(\sinh^2 u+\sin^2 v)^{1/2}} \left\{ \frac{\cosh u (\sin^2 v-\sinh^2 u)}{(\sinh^2 u + \sin^2 v)^2} \right\} \bar{e}_2$$

$$-C_1 \frac{\operatorname{Cosech} u . \operatorname{Coth} u . \operatorname{Cosec} v}{a (\sinh^2 u+\sin^2 v)} \bar{e}_3 \qquad \ldots . (33)$$

It is clear from equation (33) that flow (10) is not irrotational. For the flow (22)

$$\bar{G} = 0.$$

Hence the flow (22) is irrotational throughout.

Similar conclusions can be drawn for the other flow discussed earlier.

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EFFECT OF NON-HOMOGENEITY OF CRITICAL LOAD IN BUCKLING OF A BAR UNDER DISTRIBUTED AXIAL LOAD

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ABSTRACT

Critical load for buckling of a bar of non-homogeneous material has been obtained by two methods : (i) by using the differential equation of the deflection curve and (ii) by the energy method. The non-homogeneity of the bar has been assumed to be arising out of the variation of Young's modulus of the bar with the longitudinal distance. Critical loads obtained by the methods are found to be nearly same. The effect of non-homogeneity of this type has been observed to increase the critical load of the associated homogeneous case.

1. INTRODUCTION

The beam problem is very well-known problem in the theory of elasticity and it has significant importance in engineering structures. Thus from structural point of view, knowledge of critical buckling load of such beams is essential. The critical loads for buckling for a beam of homogeneous material under different conditions have been obtained by many investigators (Timoshenko and Gere [2]). But if the material is non-homogeneous a complete solution of the governing differential equation may not always be

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possible and in such a case approximate methods e.g. energy method, other variational methods or purturbation method, are to be used to get approximate critical load for buckling.

Our present investigation aims at the effect of non-homogeneity on critical load for buckling of a bar whose Young's modulus varies Linearly with its longitudinal distance. The Origin of co-ordinates has been suitably chosen in order to simplify the governing differential equation of the deflection curve. Solving this differential equation and using necessary and conditions critical load for buckling has been obtained. Critical load for buckling has also been obtained by assuming an approximate shape of the deflection curve and then by using the energy method. As is expected, (Timoshenko [1]) the critical load found by the energy method is slightly grater than that found by the first method. Of course, by increasing the number of parameters in the approximate equation for the deflection curve the difference of two critical load values may be made sufficiently small. In this paper it has been observed that critical load in the non-homogeneous case is greater than that in the associated homogeneous case.

2. FORMULATION OF THE PROBLEM

Let us consider a bar of length l whose lower end is vertically built in, upper end is free and the weight is uniformly distributed along the length.

We choose the co-ordinate axes as shown in the figure. If the bar buckles as shown by the dotted line in the figure, the differential equation of the curve is

$$EI \frac{d^2y}{dx^2} = \int_x^b q\,(\eta - y)\,d\xi \qquad \ldots (1)$$

where $b = a + \ell$, $a = OA$ (in the figure), and the integral on the right hand side of (1) represents the bending moment of any cross section fg produced by the uniform distributed load of intensity q.

Differentiation of (1) with respect to x gives

$$\frac{d}{dx}\left[EI\,\frac{d^2y}{dx^2}\right] = -q\,(b-x)\,\frac{dy}{dx} \qquad \ldots (2)$$

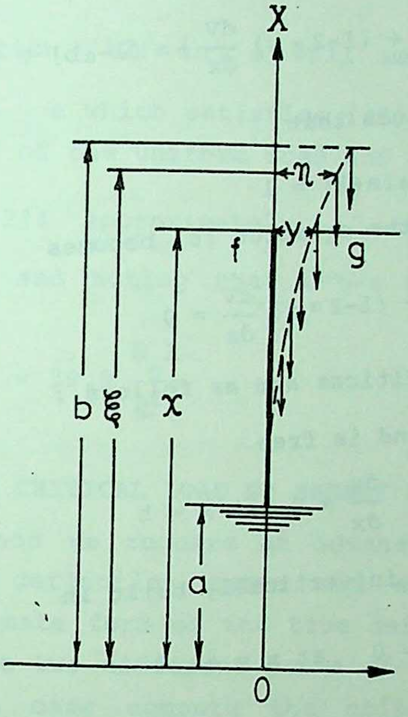

Fig.1

We now assume that the Young's modulus E of the bar obeys the law

$$E = E_o\left(\frac{X}{a}\right),\ a \le x \le b \qquad \ldots (3)$$

Where E_o is the value of E at the lower end of the bar and a is a suitable length to be defined latter.

Using (3) and writing $z = \frac{dy}{dx}$, equation (2) becomes where

$$x \frac{d^2z}{dx^2} + \frac{dz}{dx} + \alpha^2 (b-x) z = 0 \qquad \ldots (4)$$

$$\alpha^2 = \frac{qa}{E_o I} \qquad \ldots (5)$$

Setting $z = e^{-\alpha x}.V$, equation (4) becomes

$$x \frac{d^2V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} - \alpha (1-\alpha b) V = 0 \qquad \ldots (6)$$

we now choose a such that

$$\alpha b = \alpha(a+\ell) = 1$$

with this choice the equation (6) becomes

$$x \frac{d^2V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} = 0 \qquad \ldots (8)$$

The boundary conditions are as follows :

Since the upper end is free

$$\frac{d^2V}{dx^2} = \frac{dz}{dx} = 0 \quad \text{at } x = b$$

and the lower end is vertically built in

$$\frac{dy}{dx} = z = 0 \quad \text{at } x = a \qquad \ldots (9b)$$

3. SOLUTION OF THE PROBLEM

The solution of (8) is

$$V = A\,H(x) + B$$

Where $H(x) = \int \frac{e^{2\alpha x}}{x} dx$

and A and B are constants.

Thus $z = e^{-\alpha x}.\,[A\,H(x) + B]$ ... (10)

Using boundary condition (9a) we get

$$z = A e^{-\alpha x}[H(x)-H(b)+e^2] \qquad \ldots (11)$$

Where (7) has been used.

Condition 9(b) gives

$$\int_a^b \frac{e^{2\alpha x}}{x}\, dx = e^2$$

which by (7) becomes

$$\int_{2\alpha a}^{2} \frac{e^t}{t}\, dt = e^2 \qquad \ldots (12)$$

The roots of equation (12) in α a will make z=0 at x=a

The lowest value of a which satisfies (12) corresponds to the critical value of the uniform load and is found to be

$$\alpha a = 0.0234 \text{ approximately.}$$

Using (7) and (5) and noting that ℓ=b-a we compute the critical load as

$$(q\ell)_{cr} = 39.8 \frac{E_o I}{\ell^2} \qquad \ldots (13)$$

4. COMPUTATION OF CRITICAL LOAD BY ENERGY METHOD

In this method we require an advance knowledge of the shape of the deflection curve and so we have to consider an approximate form of the true deflection curve. Here we shall assume two different forms of the deflection curve and in each case compute the critical load for buckling. Since the critical load for true deflection curve is minimum, comparing the above two values of critical loads we can select the better of the two approximate curves.

Let the approximate deflection curve be y = f (x) Here we consider the following two forms of f(x).

4.1 f(x) is a Tribonometric Function of x

Here we approximate the true deflection curve by taking

$$y = \delta_1(1-\cos\frac{\pi}{2}x) + \delta_2(1-\mathrm{Cos}\frac{3\pi}{2}x) + \delta_3(1-\mathrm{Cos}\frac{5\pi}{2}x) \qquad \ldots (14)$$

where $(b-a)\,X = x-a$ ... (15)

and δ_i $(i = 1,2,3)$ are parameters.

The deflection y in (14) then satisfies the end conditions (9a) and (9b).

The bending moment at any cross section fg in the figure is

$$M = \int_x^b q(\eta-y)\,d\xi = q(b-a)\sum_{r=1}^{3}\delta_r S_r$$

Where $S_r = (1-x)\,\mathrm{Cos}\,(\theta_r X) + \frac{1}{\theta_r}[(-1)^r + \mathrm{Sin}(\theta_r X)]$

and $\theta_r = \frac{\pi}{2}(2r-1)$

The strain energy of bending is given by [Timoshenko and Gere [2] (1961), p.105]

$$\Delta U = \int_a^b \frac{M^2}{2EI}\,dx = \frac{q^2(b-a)^3}{2E_oI}\sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s$$

Where $I_{rs} = \int_o^1 S_r S_s\,F(x)\,dx$ ... (16)

and $F(x) = 1/[1+(b/a - 1)\,x]$

In view of (14) the total work produced by the load during buckling is, [Timoshenko and Gere [2] (1961), p.108]

$$\Delta T = \frac{1}{2}q\int_a^b (b-x)\left(\frac{dy}{dx}\right)^2 dx = \frac{\pi^2}{8}\,q\sum_{r,s=1}^{3}\beta_{rs}\,\delta_r\,\delta_s \,.. \qquad (17)$$

where $\beta_{rr} = \frac{r^2}{4} - \frac{1}{\pi^2}$, $r = 1,2,3$

$$\beta_{12} = \frac{3}{\pi^2}\ , \ \beta_{13} = -\frac{5}{\pi^2}\ , \ \beta_{23} = \frac{15}{\pi^2}\ , \ \beta_{rs} = \beta_{sr}$$

The critical value of the axial load satisfies the equation, [Timoshenko and Gere [2] (1961), p.84]

$$\Delta U = \Delta T \qquad \ldots (18)$$

Substituting (16) and (17) in (18) we get

$$(q)_{cr} = P\Big(\sum_{r,s=1}^{3} \beta_{rs}\,\delta_r\,\delta_s\Big) \Big/ \sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s \qquad \ldots (19)$$

where $P = \dfrac{\pi^2 E_o I}{4\ell^3}$

We now choose parameters δ_r (r=1,2,3) such that $(q)_{cr}$ becomes minimum. The conditions for that are

$$\frac{\partial}{\partial \delta_r}(q)_{cr} = 0,\ r=1,2,3. \qquad \ldots (20)$$

From (19) and (20) and from the fact that the possibility of buckling occurs when δ_r (r=1,2,3) $\neq 0$, we get

$$\det(a_{rs}) = 0,\ r,\ s = 1,2,3 \qquad \ldots (21)$$

where

$$a_{rs} = \beta_{rs} - \frac{(q)_{cr}}{p} I_{rs},\ r,s = 1,2,3. \qquad \ldots (22)$$

Now using (7) we find that

$$\frac{(q)_{cr}}{P} \quad \frac{4(u-1)^3}{\pi^2 u^2} = h(u) \qquad \ldots (23)$$

where $u = \dfrac{b}{a}$

Substituting (23) in (22) we get a_{rs} as functions of b/a. Equation (21) then gives the value of b/a corresponding to the critical load.

Solving (21) we obtain

$$\frac{a}{b} = 0.0228 \text{ approximately,}$$

Using (7) and noting that ℓ = b-a we compute the critical load as

$$(q\ell)_{cr} = 40.92 \frac{E_o I}{\ell^2} \qquad \ldots (24)$$

Comparing (13) with (24) we find that with the approximate deflection curve given by (14) the error in the value of the critical load by the energy method is less than 2.8 percent. It may be mentioned here that if we put $\delta_3 = 0$ in (14) or in otherwise if there are only two parameters in the deflection curve the error in the critical load was observed to be 7.26 percent. Instead of (14) had we assumed

$$Y = \sum_{k=1}^{n} \delta_k [1 - \mathrm{Cos}\ (2k-1)\frac{\pi}{2} x]$$

we could have obtained a value for the critical load very close to that given by (13) for sufficiently large value of n.

4.2 f(x) is a Polynomial in x.

Here the deflection curve is assumed as

$$Y = \delta_1 x + \delta_2 x^2 + \delta_3 x^3 \qquad \ldots (25)$$

So that the end conditions (9a) and (9b) are satisfied.

Using (25) and proceeding as in 4.1 we obtain equation (19), in which

$P = 288\, E_o I\, k^5/\ell^3,$

$k = (b/a)-1,$

$$I_{11} = 36h_1,$$

$$I_{22} = [16(k-z)^2 h_1 \pm 64(k-z)h_2 \pm 64h_3]/k^2,$$

$$I_{33} = [\ 9(k^4-4k^3 + 10k^2 -12k+9)h_1 + 36(k^3-5k^2+9k-9)h_2$$
$$+ 18(5k^2-18k + 27)h_3 + 54(k-6)h_4+81h_5/k^4,$$

$$I_{12} = [24\ (k-2)h_1 + 48h_2]/k,$$

$$I_{23} = [12\ (k^3-4k^2 + 7k-6)h_1+ 24\ (2k^2-7k+9)h_2$$
$$+ 12(7k-18)h_3 + 72h_4]/k^3,$$

$$I_{31} = 18\ [(k^2-2k+3)h_1 + (2k-6)h_2 + 3h_3]/k^2,$$

$$\beta_{11} = \frac{1}{4},\ \beta_{22}= \frac{1}{6},\ \beta_{33} = \frac{3}{20},\ \beta_{12} = \frac{1}{6},$$

$$\beta_{23} = \frac{3}{20},\ \beta_{13} = \frac{1}{8},\ \beta_{rs} = \beta_{sr} \text{ and } I_{rs} = I_{sr} = (r,s=1,2,3)$$

$$h_1 = [u^4(12 \log u + 23) - 48ku^3-36u^2+16u-3)/12,$$

$$h_2 = [-4u^5+5ku^4+10u^3-10u^2+5u-1]/5,$$

$$h_3 = [u^6-15u^4 + 40u^3-45u^3 + 24u-5]/30,$$

$$h_4 = [u^7-35u^4 + 105u^3-126u^2 + 70u - 15]/105,$$

$$h_5 = [u^8-70u^4 + 224u^3-280u^2 + 160u - 35]/280.$$

Now, choosing papameters δ_r $(r=1,2,3)$ such that $(q)_{cr}$ is minimum and using similar arguments as in the previous case we obtain the value of a/b corresponding to the critical load as

$$\frac{a}{b} = 0.0210 \text{ approximately.}$$

Then by (7) the critical load is obtained as

$$(q\ell)_{cr} = 44.59\ \frac{E_o I}{\ell^2} \qquad \text{... (26)}$$

Comparison of equation (26) with (24) and (13) suggests that equation (14) gives a closer approximation to the true deflection curve than that given by equation (25). Moreover, the approximate curve with two parameters ($\delta_3=0$) in (14) is even better than that with three parameters in (25).

5. CONCLUSION

The critical load for buckling in the associated homogeneous case is given by [Timoshenko and Gere, [2] p.103]

$$(q\ell)_{cr} = 7.837 \frac{E_o I}{\ell^2}$$

Comparison of this value with out present value given by (13) shows that the effect of non-homogeneity as given by (3) is to increase the critical load to about 5.08 times.

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GEOPHYSICAL PROSPECTING WITH LIGHTNING RADIATION

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ABSTRACT

A new method of geophysical prospecting using the lightning radiation is proposed. The detailed theoretical formulation shows that the ground conductivity profile can be plotted by recording the values of ratio of vertical to horizontal component of electrical field radiated from nearby lightning discharge (at a distance of 50 to 100 kms). This profile may be used for miniral exploration.

INTRODUCTION :

Use of natural fields for determination of ground conductivity dates back to 1953 when Cagniard [1] showed how the ratio of alternating magnetic field to alternating electric field plotted as a function of frequency can yield information about the variation of resistivity with depth within the earth. The method known as MT method measures simultaneously the magnetic and electrical oscillations caused by ionospheric currents and thunderstorms. These fields are recorded at different frequencies in the range from less than one hertz to 10 hertz. The same method using higher frequencies in audio range is known as AMT method.

Another closely related method was introduced in 1958 when Ward [2] measured the tilt of magnetic component of natural electromagnetic field of distant thunderstorms (at thousand of kms). At these large distances the field propagates in such a way that electric components are almost vertical and magnetic components almost hiorzontal. Presence of a conducting zone with

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a vertical boundary tilts the plane of polarisation and produces an appreciable vertical component of magnetic field. The tilt in (inclination of) the resultant magnetic field is measure of discountinuity in resistivity. This is known as AFMAG method as it uses the characteristics of audio fequency magnetic fields.

Another method on similar principle but using an artificially transmitted electromegnetic radiation is conductivity determination by wave-tilt measurements [3] where conductivity is determined from the ratio of horizontal component (directly measured) and vertical component of electrical field (determined by measuring horizontal magnetic field) emitted from VLF transmitter at a known frequency, power and distance from point of observation. Recently artificial source has been used for AMT method also [4].

These methods determine the ground conductivity and the data is interpreted according to required applications, e.g. search of metals and minirals or finding the depth of bedrock or in geothermal exploration. The first of these methods uses radiation from the source at unknown distance. The magnitude of the variation is indefinite. For long distance sources high sensitivity is needed. A near thunderstorm would disturb the conditions and even the instrument may be damaged. Moreover, the theory is not applicable to for the radiation from near thunderstorm or transmission line. This condition is also same for AFMAG method, though this method is more qualitative than quantitative. The wave-tilt measurements from artificial transmitter at known distance frequency and power are better but they too may be disturbed by thunderstorms and the theory uses many approximations [5].

In this paper a possibility of new method of geophysical prospecting using natural source is analysed and its use for different purposes is proposed.

LIGHTNING AS NATURAL TRANSMITTER :

Lightning is a heavy current transient electric discharge between two opposite charge centres within a thunderstorm or between lower charge of a thunderstorm and the ground. It results in a transfer of charge from one place to another in form of heavy currents with peak value of order of few tens of killo amperes. This time varying current through air column flows for finite time (∿ 200 μs) and produces an oscillating dipole consisting of the current channel of varying length and its image in finitely conducting ground. Experimentally observed radiation from lightning extends over whole range of electromagnetic radiation beginning from few hertz in ELF to UHF and microwave besides the visible and infrared radiation. The maximum power in radio wave range is in VLF region and peaks between 5 kHz and 10 kHz. Thus the lightning is a strong natural source of VLF radiation. This VIF radiation can be used for geophysical prospecting if the details of the influence of ground conuctivity on the radiated fields is worked out. This is discussed in next section.

GROUND CONDUCTIVITY DEPENDENCE OF LIGHTNING RADIATION :

Pathak et al. [6] showed that the VLF radiation from lightning may be accounted for by radiation emitted from an oscillating dipole consisting of lightning channel and its image in the ground. They calculated the electric and magnetic field components by formulating a vector potential associated with lightning channel and an image vector potential associated with the image of the channel. The image vector potential and therefore the resultant vertical fields are functions of ground conductivity. It can be shown [7] that the radiation from a return stroke (vertical channel from ground to cloud) is independent of ground conductivity while radiation from other strokes like K-changes (vertical or slightly inclined channel within the cloud) or horizontal discharges depends largely on ground conductivity. Physically these results are understood from the following well known facts: (i) A current channel (with any arbitrary orientation) over a perfectly conducting ground (hypothetical case) produces only vertical electrical fields (ii) An imperfectly conducting ground develops a horizontal component of electric field in two ways : (a) Horizontal currents perpendicular to the plane of incidence (a vertical plane containing source and receiver) produce a horizontal E perpendicular to the plane of incidence. (b) Vertical currents at sufficiently great heights and horizontal currents in the plane of incidence produce a horizontal E parallel to the plane of incidence. Thus the return stroke does not produce horizontal electric component and therefore the vertical component from return stroke remains independent of ground conductivity. In other cases like K-change and horizontal channel, a horizontal component is developed because of penetration of a part of energy into the earth and making reflection coefficient less than unity. As the reflection coefficient, hence the penetration depends upon the ground conductivity, the horizontal and vertical E field also become the functions of conductivity.

To obtain the analytical expressions for vertical and horizontal electric fields a vector potential **A** is formulated from the channel current I as follows:

$$A(t)=\frac{\mu_o}{4\pi}\int\frac{J(t')}{r}dv' = \frac{\mu_o}{4\pi}\int\frac{I(t')}{r}\mathbf{dz}' = \frac{\mu_o}{4\pi}\frac{o}{r}F(t') \quad ...(1)$$

where t' $(= t - \frac{r}{c})$ is rerarded time, **dz**' is retarded elemental length along the channel and A_o a vector constant in time. The distance of observation point being larger than the channel dimensions is kept outside the integral. Assuming the origin of coordinate system at starting point of channel and z-axis vertically upward r, θ and ϕ- components of **A** can be written and electric and magnetic fields can be devaluated from the component equations of the relations

$$H = \frac{1}{\mu_o} \nabla \cdot X\, A \text{ and } E = \frac{1}{\varepsilon_o} \int \nabla X H\, dt \qquad \ldots (2)$$

Similarly the field components due to image vector potential $A' = R_c A$, R_c being reflection coefficient, can be written and expressions for corresponding field components E_r'. E_θ'. E'_Φ can be evaluated. But we need most commonly measured vertical and horizontal components of the fields. So the vertical electric field is obtained by summing all the vertical components of E_r, E_θ and E_r', E_θ', i.e.

$$E_v(t) = \frac{A_o}{4\pi\varepsilon_o} \left[\frac{f_1}{r^3} \int F(t')dt + \frac{f_1}{r^2 c} \frac{\partial F(t')}{\partial t} \right] \qquad \ldots (3)$$

For horizontal electric field, the horizontal components of E_r, E_θ, E_r, E_ϑ are added to obtain the field in the plane of incidence. The field perpendicular to the plane of incidence is the sum of E_ϕ and E_ϕ'. The resultant horizontal field is square root of the sum of the squares of fields in two perpendicular horizontal directions i.e.

$$E_h(t) = \frac{A_o}{4\pi\varepsilon_o} \Big[\Big\{ \frac{f_3}{r^3} \int F(t')dt + \frac{f_3}{r^2 c} F(t') + \frac{f_4}{rc^2} \frac{\partial}{\partial t} F(t') \Big\}^2$$
$$+ \Big\{ \frac{f_5}{r^3} \int F(t')dt + \frac{f_5}{r^2 c} F(t') + \frac{f_5}{rc^2} \frac{\partial}{\partial t} F(t') \Big\}^2 \Big]^{\frac{1}{2}} \qquad \ldots (4)$$

where,

$$f_1=\cos\theta_1[2-\sin^2\theta+(3\sin^2\theta-2)R_v]-\sin\theta_1\sin\theta\cos\theta\cos(\phi_1-\phi)(1-3R'_v) \qquad \dots (5)$$

$$f_2=(1+R_v)[\cos\theta_1\sin^2\theta+\sin\theta_1\sin\theta\cos\theta\cos(\phi_1-\phi] \qquad \dots (6)$$

$$f_3=(1+3R_v)\cos\theta_1\sin\theta\cos\theta+\sin\theta_1[2-\cos^2\theta+(2-3\cos^2\theta)R_v]\cos(\phi_1-\phi) \qquad \dots (7)$$

$$f_4=(1-R_v)[-\cos\theta_1\sin\theta\cos\theta+\sin\theta_1\cos^2\theta\cos(\phi_1-\phi)] \qquad \dots (8)$$

$$f_5=-(1+R_h)\sin\theta_1\sin(\phi_1-\phi) \qquad \dots (9)$$

$$R_v=\frac{S-T}{S+T};\ R_h=\frac{\cos\psi-T}{\cos\psi+T} \qquad \dots (10)$$

with

$$S=[(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0]\cos\psi;\ T=[(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0-\sin^2\psi]^{\frac{1}{2}} \qquad \dots (11)$$

and $\psi=\pi-\theta$. The reflection coefficients R_v and R_h are the functions of permittivity and conductivity of the ground. Therefore the electric field components are also the functions of these parameters.

THE PARAMETER TO BE MEASURED

Equations (3) and (4) give time-dependent values of vertical and horizontal electric field. These can be subjected to Fourier transformations to get the frequency dependent values as

$$E_{v,h}(w) = \int_o^\infty E_{v,h}(t)e^{-iwt}dt \qquad ...(12)$$

However, the field-components thus obtained are not the functions of w and σ only. They also contain lightning parameters through A_o. Eq(1) shows that A_o would contain lightning constants like peak value of current and decay constants. These vary from one discharge to another and not easy to determine. Simple way to get rid of these parameters is to consider ratio $R = E_v/E_h$ which would depend only on orientation of lightning discharge besides w and σ. Influence of orientation of the channel is very easy to visualise through eqs. (5)-(9), as follows:

For vertical discharge $\theta_1 = 0$ then

$$f_1^v = 2 - \sin^2\theta + R_v(3\sin^2\theta - 2)$$

$$f_2^v = (1+R_v)\,\text{Sin}^2\theta$$

$$f_3^v = (1+3R_v)\,\text{Sin}\theta\cos\theta \qquad ...\ (12)$$

$$f_4^v = -(1-R_v)\sin\theta\cos\theta$$

$$f_5^v = 0$$

For a vertical discharge starting from ground like return stroke; $\theta = \pi/2$, then $f_1 = f_2 = (1+R_v)$, $f_3 = f_4 = f_5 = 0$. This gives $E_h^v = 0$ and $R = \infty$ in accordance with the concepts that a vertical current near ground does not produce any horizontal field. Also for this case $|R_v| = 1$, so E_v becomes independent of ground conductivity which is obvious when the horizontal component is not developed i.e. energy ponetration into the ground is zero. Same result is obtained by Pathak and Hazarika [7].

For vertical discharge at sufficient height h above the ground $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The computed values of R at a fixed frequency of 5 kHz are plotted with ground conductivity for different values of r in Fig.1a.

For horizontal discharge; $\theta_1 = \pi/2$. But the channel may have any arbitrary inclination from the plane of incidence. If the channel is perpendicular to the plane of incidence i.e. $_1 - \; = \pi/2$ so that $f_1 = f_2 = f_3 = f_4 = 0$ and $f_5^h = (1 + R_v)$, we have $E^h = 0$ and therefore $R = 0$. The field is confined to a plane perpendicular to the plane of incidence. For the case when channel is in the plane of incidence. $\phi_1 - \phi = 0$, so that

$$
\begin{aligned}
f_1^h &= (1 - 3R_v) \sin\theta \cos\theta \\
f_2^h &= (1 + R_v) \sin\theta \cos\theta \\
f_3^h &= 2 - \cos^2\theta + (2 - 3\cos^2\theta) R_v \qquad \ldots (13) \\
f_4^h &= (1 - R_v) \cos^2\theta \\
f_5^h &= 0
\end{aligned}
$$

This gives E_h^h confined to the plane of incidence as expected. Again $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The variation in the values of R is too small to be plotted or to be measured over the experimental error (see Table 1). The overall variation of R for arbitrary orientation of horizontal channel is between zero and 0.41. For requirement of high sensitivity and of determination of orientation $(\varphi_1 - \varphi)$, the horizontal channel is not recomended for use in determination of ground conductivity.

FIELD PROCEDURE :

As shown above the parameter depending upon minimum number of lightning parameter is ratio R of vertical and horizontal electric field. A non-zero, finite value of R with sufficient variation is obtained only for the case of intracloud vertical lightning. Field procedure would be to measure E_v and E_h at a fixed frequency (say 5 kHz). These are measured with the help of vertical and horizontal whip antinnas suplimented by tuned amplifiers. Amplification is to be sufficient to amplify a signal of a fraction of mV to the magnitude required by recorder. For measuring the resultant horizontal field, signals from two crossed whips are fed into a system of squaring, adding and square-rooting IC's connected as shown in Fig.1b. After this stage E_v and E_h may be recorded separately and ratio may be obtained manually or a dividing IC may be used to obtain R directly. The distance of discharge is recorded from the time log Δt between electrical signal and thunder heard. Then $r = \Delta t \times$ speed of sound. If $E_v > E_h$ i.e. $R > 1$, the value of σ may be determined from Fig. 1a. By using many amplifieres tuned to different frequencies, average σ for different depths of penetration for these frequencies is plotted. This gives the conductivity profle with depth. This may be used for exploring the presence of minirals within the earth, and the distribution of ground water.

CONCLUSION :

From the fargoing discussion it is clear that by the observation of ratio of vertical and horizontal electic field radiated from nearly lightning (in range of distance of 50 to 100 km) the ground conductivity profile may be determined for the use in miniral exploration. The expressions are not valid for very near large distances (more than 100 km).

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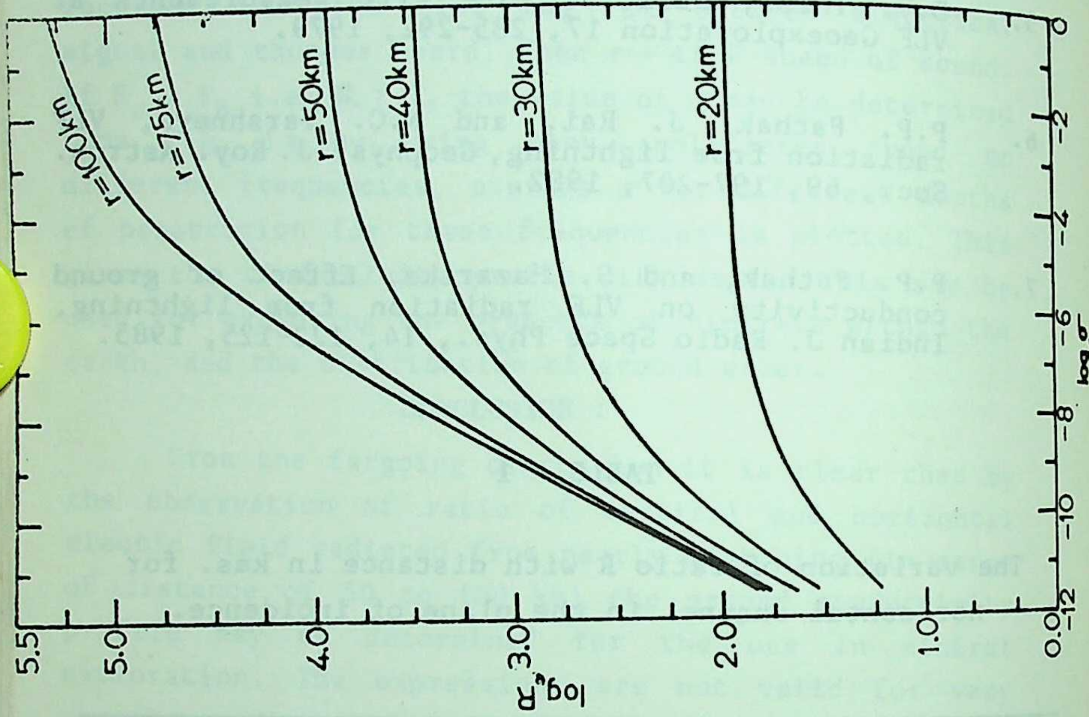
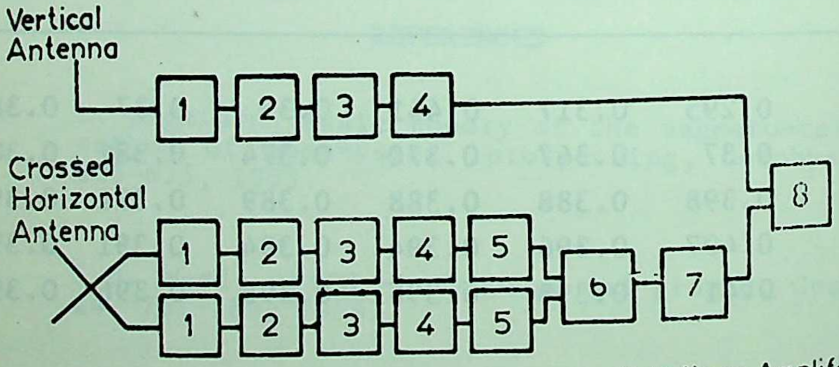
TABLE - 1

The variation of ratio R with distance in kms. for horizontal channel in the plane of incidence.

$\sigma(\Omega\,\mathrm{m})^{-1}$	20	30	40	50	75	100
10^{-5}	0.295	0.317	0.441	0.35	0.37	0.38
10^{-4}	0.37	0.367	0.370	0.374	0.381	0.385
10^{-3}	0.398	0.388	0.388	0.389	0.391	0.393
10^{-2}	0.407	0.396	0.394	0.394	0.391	0.396
10^{-1}	0.41	0.398	0.396	0.395	0.396	0.397




Vertical Antenna
1
2
3
4
8
Crossed Horizontal Antenna
1
2
3
4
5
1
2
3
4
5
6
7

1. Matching Amplifier (Unity gain), 2. Tuner, 3. Voltage Amplifier, 4. Power Amplifier, 5. Squaring V.A., 6. Adding Circuit, 7. Square Rooting V.A., 8. Dividing Circuit (Digital Recorder)

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NOTE ON THE DEFLECTION IN A SIMPLY SUPPORTED TIMOSHENKO BEAM OF NONHOMOGENEOUS MATERIAL

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ABSTRACT

In expressions for the deflections in a simply supported beam of nonhomogeneous material have been obtained by using both technical and Timoshenko beam theories. It has been assess that the Young's modulus of the beam is a function of position and the beam is under the action of variable load. A comparative study has been made, to assess the effect of nonhomogeneity as has been considered in our note, in the tables for different values of the nonhomogeneity parameter.

Keywords : Timoshenko beam, Deflection, Nonhomogeneous material, Young's modulus.

INTRODUCTION :

The one dimensional theory of beams may have wider range of applicability provided the effects of transverse shear deformation, and the effects of rotatory inertia in the case of vibrating beams, are considered. When these effects are taken into account in the technical theory of beams, the new theory is known as Timoshenko beam theory [17, 18]. Dengler and Goland [6], Goland et

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al.[8], Miklowitz [11,12] studied the problems of dynamic response of Timoshenko beams. Achenback and Sun [1] obtained the steady state solutions of the response of the Timoshenko beam to a force moving with constant velocity over the beam which was supported by a deformable foundation. Ahmadi and Hashemi [2] studied the problem of vibration of an infinitely long Timoshenko beam under two types of random loadings. Chang and Juan [3] derived a set of equations for the free vibration of an inclined bar with an end constraint including the effect of rotatory inertia and shear deformation. Chen [4] presented the general dynamic stiffness matrix of a Timoshenko beam for transverse vibration. Among the many other problems of the dynamic response of Timoshenko beams mention may be made of the interesting works of Howson and Williams [9], Mead [10], Nayfeh [13], Park [14], Rao [15], Rossettos and Squires [16]. The statical Timoshenko beam problems have also received considerable attention in the literature. A comparative study, for the deflections in a uniformily loaded simply supported elastically homogeneous beam following the above theories, has been given in Dym and Shames [7].

As we know that there are plenty of materials in nature which are not elastically homogeneous and as it has been experimentally verified that in most of the cases the elastic co-efficients should not be taken as uniform throughout the beam, the purpose of the present investigation is to extend the foregoing studies given in Dym and Shames [7] to a class of nonhomogeneous beams subjected to nonuniform loading. The nonhomogeneity has been supposed to be arising from the variation of

Young's modulus with position. The general expressions for the deflections have been given. Numerical evaluations of deflections at different positions of the beam have been carried out to study the effects of shear deformation at a glance. Results of the corresponding homogeneous cases indicate that nonhomogeneity has significant effect on deflection.

FORMULATION OF THE PROBLEM AND ITS SOLUTION

(a) Solution based on the technical theory of beams

Choosing x-axis along the axis of the beam, let the position of the beam be given by $a \leqslant x \leqslant 2a$ ($a > 0$). The equation of equilibrium for the bending of the beam is given by

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = qx^2 \qquad \ldots (1)$$

where w represents the deflection of the centreline of the beam under the action of variable load q(x) on the centroidal axis of the beam. E is the variable Young's modulus and I is the moment of inertia of the cross section of the beam about an axis lying in its plane.

We assume that the Young's modulus of the beam is a function of position according to the law,

$$E = E_o\, e^{-m(\frac{x}{a})} \qquad \ldots (2)$$

and loading depends upon the position following the relation,

$$q = q_o \left(\frac{x}{a}\right)^n \qquad \ldots (3)$$

where E_o, q_o, m and n are real constants.

As regards the boundary condition of the problem we assume that both ends of the beam are simply supported. Mathematically speaking,

$$w = 0$$

$$\text{and} \quad \frac{d^2w}{dx^2} = 0 \text{ at } x = a \text{ and } x = 2a. \qquad \ldots (4)$$

Introducing the nondimensional variable $X = \frac{x}{a}$ and taking into account of (2) and (3), field equation (1) becomes,

$$\frac{E_o I}{a^4} \frac{d^2}{dX^2}\left[e^{-mX} \frac{d^2w}{dX^2}\right] = q_o X^n \qquad \ldots (5)$$

and the boundary condition (4) takes the form,

$$w = 0$$

$$\text{and} \quad \frac{d^2w}{dX^2} = 0 \text{ at } X = 1 \text{ and } X = 2. \qquad \ldots (6)$$

Thus our problem is to solve (5) subject to (6).

Integrating equation (5) twice with respect to X and making use of the boundary condition $\frac{d^2w}{dX^2} = 0$ at $X = 1$ and $X = 2$ we obtain,

$$\frac{d^2w}{dx^2} = A_1 X^{n+2} e^{mX} + B_1 X e^{mX} + C_1 e^{mX} \qquad \ldots (5a)$$

finally integrating (5a) twice with respect to X and making use of the boundary condition $w = 0$ at $X = 1$ and $X = 2$ we obtain the expression for w as

$$w = w_o(X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) \qquad \ldots (7)$$

where

$$f(X) = \int_1^X \left(\int_1^u t^{n+2} e^{mt} dt \right) du$$

$$g(X) = \int_1^X \left(\int_1^u t\, e^{mt} \, dt \right) du$$

$$\theta(X) = \int_1^X \left(\int_1^u e^{mt} dt \right) du$$

$$A_1 = \frac{a^4}{E_o I} \, \frac{q_0}{(n+1)\quad(n+2)}$$

$$B_1 = A_1 (1 - 2^{n+2})$$

$$C_1 = A_1 (2^{n+2} - 2)$$

and w_o is the value of $\frac{dw}{dX}$ at $X = 1$ and it satisfies the equation

$$w_o + A_1 f(2) + B_1 g(2) + C_1 \theta(2) = 0. \qquad \ldots (8)$$

(b) Solution based on the Timoshenko beam theory

The method discussed in the previous section does not indclude the effects of shear deformation. For short stubby beams, however, it is natural that this contribution cannot be neglected. Hence arises, the question of considering Timoshenko theory of beams. In this theory it is assumed that line elements normal to the centreline of the beam in the undeformed state move only in the vertical direction and also remain vertical during deformation. Line elements tangent to centreline undergo a rotation β (x) and satisfy the relation,

$$\frac{dw}{dx} = \Psi(x) + \beta(x), \qquad \ldots (9)$$

$\frac{dw}{dx}$ being the total slope of the centreline and $\Psi(x)$ is the rotation of the line elements due to bending only.

In this case we get the two field equations as,

$$\frac{d}{dx}\left[EI\frac{d\Psi}{dx}\right] + KGA\left(\frac{dw}{dx} - \Psi\right) = 0 \qquad \ldots (10)$$

and $$\frac{d}{dx}\left[KGA\left(\frac{dw}{dx} - \Psi\right)\right] + q = 0 \qquad \ldots (11)$$

where G is the shear modulus, K is the shear coefficient and A is the area of cross section.

Differentiating (10) with respect to x and using (11) we get the single field equation as

$$\frac{d^2}{dx^2}\left[EI\,\frac{d\Psi}{dx}\right] = q. \qquad \ldots (12)$$

In terms of dimensionless variable $X = \frac{x}{a}$ the field equation (12) when (2) and (3) are taken into account, becomes ,

$$\frac{E_o I}{a^3} \frac{d^2}{dX^2} \left[e^{-mX} \frac{d\Psi}{dx} \right] = q_o X^n \qquad \ldots (13)$$

and the boundary conditions become

$$w = 0$$

and $\frac{d\Psi}{dX} = 0$ at $X = 1$ and $X = 2$. ... (14)

Following the same method as discussed in the previous section the solution of (13) satisfying (14) is obtained as,

$$w = a\,\Psi_o\,(X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) - D_1\ \Phi(X) \qquad \ldots (15)$$

where $\phi(X) = (n+2) \int_1^X t^{n+1}\ e^{mt}\ dt + (1-2^{n+2}) \int_1^X e^{mt}\ dt$

and $D_1 \qquad A_1 \frac{12+11\nu}{60} (\frac{h}{a})^2$.

In deriving (15) we have replaced G by $E/2(1+\nu)$ and A by bh. We have also introduced $I = \frac{1}{12} bh^3$, where b and h are the dimensions of the rectangular cross section of the beam and we have chosen the value of shear co-efficient K for a rectangular cross section from a paper by Cowper [5] as,

$$K = \frac{10(1+\nu)}{12+11\nu}$$

ν being the constant Poisson's ratio.

The quantity Ψ_0 appearing in equation (15) is the value of Ψ at $X=1$ and it satisfies the equation,

$$a\Psi_0 + A_1 f(2) + B_1 g(2) + C_1 \theta(2) - D_1 \phi(2) = 0. \qquad \dots (16)$$

Comparing equation (15) with equation (7) we see that the effect of shear deformation on the deflection in the nonhomogeneous beam lies in the last term in equation (15).

NUMERICAL RESULTS

To study the effect of nonhomogeneity as stipulated in (2) and the effect of nonuniform loading of the type (3) we have computed deflections for different values of m and n on taking Poisson's ratio $\nu = 1/3$. Deflection valuee have been computed at $X = 1.1\,(.1)2$ and the values of $R=\frac{E_0 I}{q_0 a} 4w$ for technical beam theory as well as for Timoshenko beam theory have been displayed in Tables 1 to 4 to consider different cases of interest. Moreover, to study the effects of shear deformation for increasing values of h/a the deflections for technical as well as shear theory have been shown for a particular position of the beam in Table 5.

Table-1

Values of R for m = 0,n = 0 and h/a = .1

X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.004087	.007733	.010587	.012400	.013020	.012400	.010587	.007733	.004087	0
R_{Timo}	.004205	.007942	.010862	.012713	.013347	.012713	.010862	.007942	.004205	0

Values of R for m = 1, h/a =.1 and different n

	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
n = 0											
R_{Tech}		.017017	.032687	.045709	.054894	.059236	.058008	.050881	.038066	.020486	0
R_{Timo}		.017453	.033510	.046860	.056295	.060791	.059601	.052369	.039277	.021216	0
n = .5	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.020659	.039745	.055704	.067078	.072600	.071316	.062743	.047069	.025387	0
R_{Timo}		.021173	.040726	.057090	.068784	.074313	.073295	.064611	.048604	.026319	0
n = 1	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}		.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
n = 1.5	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.030958	.059729	.084063	.101748	.110758	.109459	.096883	.073081	.039587	0
R_{Timo}		.031688	.061149	.086108	.104314	.113694	.112559	.099867	.075586	.041140	0
n = 2	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.038204	.073803	.104071	.126267	.137825	.136609	.121273	.091732	.049799	0
R_{Timo}		.039083	.075528	.106574	.129436	.141486	.140512	.125069	.094950	.051816	0

Table - 3

Valuesof R for $\tilde{n}$ = 1, h/a = .1 and different m

	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 0											
R_{Tech}		.006004	.011403	.015691	.018485	.019531	.018714	.016071	.011797	.006258	0
R_{Timo}		.006165	.011695	.016084	.018945	.020021	.019195	.016501	.012131	.006450	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = .5											
R_{Tech}		.012245	.023431	.032588	.038877	.041641	.040461	.035216	.026161	.014004	0
R_{Timo}		.012551	.024006	.033386	.03984	.042701	.041535	.036209	.026960	.014478	0
m = 1	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}		.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}		.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 1.5											
R_{Tech}		.052475	.101717	.144163	.176031	.193558	.193371	.173043	.131880	.072029	0
R_{Timo}		.053749	.104229	.147828	.180697	.198983	.199197	.178756	.136767	.075121	0
	X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
m = 2											
R_{Tech}		.110298	.214942	.307145	.379021	.421948	.427315	.387815	.299568	.165430	0
R_{Timo}		.113087	.220521	.315423	.389760	.434701	.441328	.401904	.311949	.173491	0

Table - 4

Value of R for m = 1, n = 1 and different h/a

X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
h/a=.1										
R_{Timo}	.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
h/a=.01										
R_{Timo}	.025227	.048604	.068267	.082423	.089486	.088156	.077795	.058523	.031634	0

Table - 5

Values of R for m = 1, n = 0, X = 1.5 and different h/a

R_{Tech} = 0.059236

h/a	R_{Timo}
0.1	0.060791
0.2	0.065458
0.3	0.073236
0.4	0.084124
0.5	0.098123

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COMMON FIXED POINT THEOREMS IN METRIC LINEAR AND UNIFORM SPACES

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In this paper we generalize and improve a fixed point theorem in normed linear spaces due to Khan, Imdad and Sessa [3] to matric linear spaces and also prove a common fixed point theorem in uniform spaces.

Let (X,d) be a metric linear space. It has ball conveity if $r \in R^+$, x, $y \in X$ with $d(x,0) = d(y,0) = r \Longrightarrow d(\frac{x+y}{2}, 0) \leq r$.

In was observed by Sastry and Naidu [6] that in a metric linear space (X,d) with ball convexity (B.C.), the function d(tx, 0) is a increasing function to t on R^+ for any x in X. We first observe the following:

LEMMA 1. Suppose (X, d) is a metric linear space with ball convesity, $\{a_n\}$ is a sequence in X and $\{\alpha_n\} \subseteq [0, 1]$ such that $\{\alpha_n\}$ is bounded away from zero. If $\{\alpha_n a_n\}$ converges to zero then $\{a_n\}$ also converges to zero.

Proof. Since $\{\alpha_n\}$ is bounded away from zero, there exists a positive integer N such that $\alpha_n > \frac{1}{N}$ for sufficiently large n. Since d(tx, 0) is an increasing function of t or R^+ for any x in X, we have $d(a_n,0) = d(\frac{1}{\alpha_n} \alpha_n a_n, 0) \leq d(N\alpha_n a_n, 0)$, for sufficiently large n,

$\leq N d(\alpha_n a_n, 0)$ since d is translation invariant.

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Since $\{\alpha_n a_n\}$ converges to zero it follows that $\{a_n\}$ also converges to zero.

Definition (Jungck [2]). Two self-maps f and g on a metric space (X,d) are said to be compatible on X iff $\underset{n\to}{\mathrm{Lt}}\, d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\underset{n\to}{\mathrm{Lt}}\, fx_n = u = \underset{n\to}{\mathrm{Lt}}\, gx_n$ for some u in X.

THEOREM 2. Let (X,d) be a metric linear space with ball convexity and K be a nonempty convex subset of X; f, g, h be self-mappings of K satisfying

$$(2.1)\quad d(fx,gy) \le \phi(d(hx,hy), d(hx,fx)+d(hy,gy), d(hx,gy) + d(hy,fx), d(hx,fx)+d(hx,gy), d(hy,gy)+d(hy,fx))$$

for all x, y in K where $\phi : R^{+5} \to R^{+}$ is an upper semi continuous function such that (i) $\phi(0, t, t, t, t) < t$ for $t > 0$, (ii) $\phi(t, 0, 2t, t, t) < t$ for $t > 0$.

Suppose there exists a sequence $\{C_n\}$ [0,1] which is bounded away from zero.

Suppose for $x_0 \in K$, there exists a sequence $\{x_n\}$ such that $hx_{2n+1} = (1-C_n)\, hx_{2n} + C_n\, fx_{2n}$, $hx_{2n+2} = (1 - C_n)\, hx_{2n+1} + C_n\, gx_{2n+1}$ for $n \ge 0$. Suppose either f and h or g and h are compatible on K. If $\{hx_n\}$ converges to z in K and h is continuous at z then z is the unique fixed point of f, g and h.

Proof:- Clearly $hx_{2n+1} - hx_{2n} = C_n\, (fx_{2n} - hx_{2n})$.

By Lemma 1, $fx_{2n} - hx_{2n} \to 0$ as $n \to \infty$ and hence $fx_{2n} \to z$ as $n \to \infty$. Similarly $gx_{2n+1} \to z$ as $n \to \infty$.

Suppose f and h are compatible on K. then

$$d(fhx_{2n}, hfx_{2n}) \to 0 \text{ as } n \to \infty .$$

Since h is continuous at z we have $hfx_{2n} \to hz$, $h^2x_{2n} \to hz$ as $n \to \infty$ and hence $fhx_{2n} \to hz$ as $n \to \infty$. Now,

$$d(fhx_{2n}, gx_{2n+1}) \leq \emptyset\,(d(h^2x_{2n}, hx_{2n+1}), d(h^2x_{2n}, fhx_{2n}) + d(hx_{2n+1}, gx_{2n+1}), d(h^2x_{2n}, gx_{2n+1}) + d(hx_{2n+1}, fhx_{2n}), d(h^2x_{2n}, fhx_{2n}) + d(h^2x_{2n}, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) + d(hx_{2n+1}, fhx_{2n})).$$

Letting $n \to \infty$, we get

$d(hz, z) \leq \emptyset\,(d(hz, z), 0, 2\,d(hz, z), d(hz, z))$.

By (ii), we get $hz = z$.

Suppose g and h are compatible on K.

Then putting $x = x_{2n}$, $y = hx_{2n+1}$ in (2.1) and letting $n \to \infty$ and using (ii) one can prove that $hz = z$.

Now,

$$(2,2)\quad d(fz, gx_{2n+1}) \leq \emptyset\,(d(hz, hz_{2n+1}), d(hz, fz) + d(hx_{2n+1}, gx_{2n+1}), d(hz, gx_{2n+1}) + d(hx_{2n+1}, fz), d(hz, fz) + d(hz, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) + d(hx_{2n+1}, fz)).$$

Letting $n \to \infty$, we get

$d(fz, z) \leq \emptyset\,(0, d(z, fz), d(z, fz), d(z, fz)\ d(z, fz))$.

By (i), we get $fz = z$.

Similarly putting $x = x_{2n}$, $y = z$ in (2.1) and letting $n \to \infty$ and using (i), one can prove that $gz = z$.

Thus z is a common fixed point of f, g and h. uniqueness of common fixed point follows easily from (2.1) and (ii).

COROLLARY 3. Theorem 2 holds if the inequality (2.1) is replaced by

(3.1) $d(fx, gy) \leq a\, d(hx, hy) + b\,[d(hx, fx) + d(hy, gy)]$

$$+ c\,[d(hx, gy) + d(hy, fx)]$$

for all $x, y \in K$ where $a + b + c \geq 0$, $b + c < 1$, $a + 2c < 1$.

Proof . Corollary 3 follows by selecting $\emptyset(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3$ in Theorem 2.

REMARK 4 . In Corollary 3, the assumption '$\{hx_n\}$ converges to z in K' cannot be dropped even when $g = f$, h = identify map, $C_n = 1$ for all n in view of the following example.

Example. Define $F ; R_+ \rightarrow R_+$ as $F(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 1, & t \geq 1 \end{cases}$

Then (R, d), where $d(x, y) = F(|x-y|)$ for all $x, y \in R$, is a Complete matric linear space with ball convexity.

Let $K = [0, \infty)$. Define $f : K \rightarrow K$ as $fx = x+1$. Then $d(fx, fy) \leq \frac{1}{2} d(x,y) + \frac{1}{4} [d(x, fx) + d(y, fy)]$ for all $x, y \in K$.

Here $a = \frac{1}{2}$, $b = \frac{1}{4}$, $c = 0$.

For any $x_o \in K$, the sequency $\{x_n\}$ defined by $x_{n+1} = fx_n$, $n = 0, 1, 2, \ldots$ does not converge to any $z \in K$.

f has no fixed point.

REMARKS. The following example shows the situation where Corollary 3 is applicable and Theorem 7 is not applicable even when $g = f$, $C_n = 1$ for all n.

Example. Let $X = K = (0, \infty)$, $d(x, y) = |x - y|$ for all $x, y \in K$. Let $fx = x^2$, $hx = 2x^2$ for $x \in K$. Clearly $f(X) = h(X)$ and $d(fx, fy) = \frac{1}{2} d(dx, hy)$ for all $x, y \in K$.

Clearly f and h are not weakly commuting. But f and h are compati

For $x_o = \frac{1}{2}$ there exist $x_1 = \frac{1}{2\sqrt{2}}$, $x_2 = \frac{1}{2^2}$, $x_3 = \frac{1}{2^2\sqrt{2}}$, ...

such that $hx_{n+1} = fx_n = \frac{1}{2^{n+2}} \to 0 \in K$.

Clearly h is not linear. Zero is the only common fixed point of f and h.

REMARKS 6. In Corollary 3, the assumption "h is continuous at $z \in K$" cannot be dropped even when $g = f$, $C_n = 1$ for all n in view of the following example.

Example . Let $X = [0,1] = K$, $d(x,y) = |x-y|$ for all $x, y \in K$. Define $fx = \frac{x}{2}$ for all $x \in K$;

$hx = x$ if $x \neq 0$, $h(0) = 1$. Here $f(X) \not\subset h(X)$, h is not linear.

Clearly $d(fx, fy) \leq \frac{1}{2}\, d(hx, hy) + \frac{1}{2}\,[d(hx,fx) + d(hy,fy)]$ for all $x, y \in K$. f and h are weakly commuting and hence compatible. For $x_o \neq 0$ there exists $\{x_n\}$ such that $hx_{n+1} = fx_n \to 0 \in K$. Clearly h is not continuous at 0. But 0 is not the common fixed point of f and h.

Corollary 3 is an improvement of the following theorem of [3].

THEOREM 7 ([3]). Let f and h be two self-mappings of a linear normed space X such that f, h is weakly commuting pair, h is continuous, linear and $f(X) \subseteq h(X)$. Further, we have

$$||fx - fy|| \leq a\, ||hx - hy|| + b[\,||hx - fx|| + ||hy - fy||\,]$$
$$+ c\,[\,||hx - fy|| + ||hy - fx||\,]$$

for all $x, y \in X$ where $a + b + c \geq 0$, $a + 2b + 2c \leq 1$. Let $x_o \in X$ and $\{C_n\}$ be a sequence satisfying $0 < C_n \leq 1$ for $n \geq 0$ and $\underset{n\to\infty}{\mathrm{Lt}}\, C_n > 0$. Suppose the sequence $\{hx_n\}$ defined by $hx_{n+1} = (1 - C_n)\, hx_n + C_n\, fx_n$ (for $n \geq 0$) converges to a point z in X. Then z is a coincidence point of f and h. If $b > 0$ then fz is the unique common fixed of f and h.

In the proof of Theorem 7, they [3] used $a \geq 0$, $b \geq 0$ without mentioning it in the hypothesis for the existence of coincidence point.

REMARK 8. Theorem 2 is true if either f and h or g and h are compatible and h is continuous at z are replaced by either $fh = hf$ or $gh = hg$ and h^k is continuous at z for some positive integer k.

One can prove the theorem in Remark 8 by the similar lines of cases (7) and (2) in Theorem (1.1 of Naidu and Prasad [5].

Now we prove a theorem in uniform spaces.

Let (X, U) be a Hausdorff sequentially complete uniform space and let P be a fixed family of pseudo-metrics on X which generates the uniformity U. For any pseudo-metric p on X and $r > 0$, let $V_{(p,r)} = \{(x,y,) \mid x,y, \in X, p(x,y) < r\}$.

Let $G = \{V_n \mid V_n = \bigcap_{i=1}^{n} V_{(pi,\ ri)},\ r_i > 0,\ p_i \in P, i = 1, 2, \ldots\ldots, n\}$.

Two self-maps f and h in (X,U) said to be compatible if $p(fhx_n, hfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence such that $\operatorname{Lt}_{n\to\infty} fx_n = u = \operatorname{Lt}_{n\to\infty} hx_n$ for some u in X where p is any pseudo-metric on X.

THEOREM 9. Let f and h be self-maps of X such that h is continuyous, $f(X) \subseteq h(X)$ and f and h are compatible on X. Also for any V_1, V_2, V_3, V_4, V_5 in G and x, y in X,

$(hx, hy) \in V_1$, $(hx, fx) \in V_2$, $(hy, fy) \in V_3$, $(hx, fy) \in V_4$, $(hy, fx) \in V_5$ implies

(9.1) $(fx, fy) \in a_1(x, y)\ V_1 \circ a_2(x,y)\ V_2 \circ a_3(x,y)\ V_3 \circ a_4(x,y)\ V_4 \circ a_5(x, y)\ V_5$

where $a_i : X \times X \to R^+$ for $i = 1, 2, 3, 4, 5$ such that

$$\sup_{x,y \in X} \sum_{i=1}^{5} a_i\,(x,y)] = \lambda < 1.$$

Then f and g have a unique common fixed point z in X.

Proof. Let x, y be arbitrary in X and V be arbitrary in G. Let p be the Minkowski pseudo-metric of V and put $p(hx,hy) = r_1$, $p(hx,fx) = r_2$, $p(hy,fy) = r_3$, $p(hx,fy) = r_4$, $p(hy,fx)=r_5$. Then with arbitary $\ \ > 0$, we have

$(hx,hy) \in (r_1 + \varepsilon)\, V$, $(hx,fx) \in (r_2+\varepsilon)\, V$, $(hy,fy) \in (r_3+\varepsilon)\, V$, $(hx,fy) \in (r_4+\varepsilon)\, V$, $(hy,fx) \in (r_5+\varepsilon)\, V$.

Write $a_i(x, y) = a_i$ for the sake of convenience.

Therefore by (9.1), we have

$$(fx,fy) \in a_1(r_1+\varepsilon)V_o\, a_2(r_2+\varepsilon)V_o\, a_3(r_3+\varepsilon)V_o\, a_4(r_4+\varepsilon)V_o\, a_5(r_5+\varepsilon)V$$

Since ε is arbitrary we have

$p(fx, fy) \le a_1\, p(hx,hy) + a_2\, p(hx,fx) + a_3\, p(hy,fy) + a_4 p(hx, fy) + a_5 p(hy,fx)$ and it follows that

(9.2) $p(fx,fy) \le \lambda \max\,\{p(hx,hy),\ p(hx,fx),\ p(hy,fy),$

$p(hx,fy),\ p(hy,fx)\}$ for all x, y in X.

For x_o in X, define $\{x_n\}$ such that $fx_n = hx_{n+1}$ for $n=0,1,2,\ldots$ It can be possible since $f(X) \subseteq h(X)$.

One can prove, by following the procedure as in Das and Naik [1] or Naidu and Prasad [4], that $\{fx_n\}$ is Cauchy. Hence $\{fx_n\}$ converges to some $z \in X$. By continuity of h and compatibility of f and h we get $fhx_n \to fx$ as $n \to \infty$.

Using (9.2) to the terms $p(fhx_n,\ fz)$ and $p(fhx_n,\ fx_n)$ and letting $n \to \infty$ we get $p(hz,fz) = 0 = p(hz,\ z)$.

So (hz, fz) and (hz, z) are in V for all V in G. Hence hz = fz = z. Uniqueness of common fixed point follows easily by (9.2).

REMARK 10. Theorem 9 is true if f and h are compatible and h is continuous are replaced by fh = hf and h^k is continuous for some positive integer k .

Remark 10 is an improvement of Theorems 2.1, 3.1, 3.2, 4.2 of [1].

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STUDIES ON SOIL CONSERVATION VALUE OF COMMON PLANTS OF SHIVALIKS

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ABSTRACT

In the present study soil conservation value of *Cynodon dactylon*, *Adhatoda vasica*, *Ipomea carnea* *Cymbopogon martimii* and *Sida sps* has been investigated and it has been found that *Cynodon dactylon*, *Adhatoda vasica* and *Cymbopogon martimii* are the suitable species for plantation on hilly slopes because thier soil conservation value is maximum, being 82.5%, 72.1% and 48.4%, respectively. The moisture content of soil which helps in formation of soil texture under these plant species was also higher being 18.5%, 20.5% and 15.2%, respectively, (Fig.1).

Key words : Soil conservation value, soil moisture, leaf area index.

INTRODUCTION

Danger is looming large in Shivalik, known as a delicate (Kuchcha) mountain range. According to the botanists the 1000 feet high mountain surrounding the famous Mansa Devi temple faces collapse in a few years as a result of induscriminate cutting of plants, pilgrims activties and uncontrolled grazing. People felled trees at Bilkeshwar too build a Bilkeshwar colony. Not only did they cut valuable tree but even diverted a stream into the Ganga making themselves susceptible to land slide and floods. These hills are surrounded by many villages. The grazing animals as well as wood and fodder collectors of these villages reduce the forest wealth. It leads to loss

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of organic layer and humus which is helpful in conserving soil moisture. It is estimated that approximately 40 crore animals grage in Indian forests. These result in soil erosion on hilly slopes in many parts of India converting them into barrenlands. Shivaliks and Aravali hills of India are facing severe ill effects of uncontrolled grazing by heeds of cattle, Maheshwari [6]. Soil erosion from Shivalik in Hardwar and river bank in downstream was found to the enormous during rains. Ambasht [1] has found the soil conservation values of *Euphorbia munja* (92-96%), *Cynodon dactylon* (90-97%) and Weed *Euphorbia hirta* (10-12%). Besides, several grasses have been found suitable for checking soil erosion in the Nilgiri hill e.g. *Eragrostis amabitis*, *E. cervula*, *Cynodon dactylon* and *Dactylis glomerata*,, Bhatia [3]. Hence, to check soil erosion, the most effective and natural process would be to plant suitable species at suitable sites, Shanker [7].

MATERIALS AND METHODS

The selection of plants has been done on the basis of their economic importance, root habit and ground cover. Plant were planted on the experimental sloping plots each of 1x3m in size (*In the G.Kangri botanical garden*) with an inclination angle of 13°. At the base of the slope cemented runoff collectors were constructed to receive the water runoff and eroded soil from the plots. Water was sprayed upon the plots from a height of 1mt for 30 min at a constant speed of 16l/min showering done at 15 days intervals, Ambasht [2].

(a) $$CV = 100-\left(\frac{S_{wp}}{S_{wo}} \times 100\right)$$

where - CV = Conservation value

S_{wp} = Weight of soil washed from vegetated plots

S_{wo} = Weight of soil eroded from the bare plot

(b) LAI = TLA/TGA

where
LAI = Leaf area index
TLA = Total leaf area
TGA = Total ground area

RESULTS AND DISCUSSION

In the present study soil conservation value and soil moisture content of *Adhatoda vasica, Ipomea carnea, dactylon, Cymbopogon martimii & Sida sps* has been investigated. As evident from table (I) *Cynodon dact-ylon* and *Adhatoda vasica* showed higher conservation value i.e. 49.2% and 45.0% the moisture content of soil, which helps in formation of soil texture under these plants was also higher being 19.6% and 21.9%, respectively.

Ipomea carnea, Sida sps and *Cymbopogon martimii* showed lower conservation value standing at 25.8%, 37.8%, 37,6% and 39.3%, respectively. On day 15 the conservation value of *Cynodon dact-ylon* and *Adhatoda vasica* was found to be 68.9% and 52.6%, respectively and the moisture content of soil stood at 16.6% and 20.2%, respectively. *Ipomea carnea* , *Sida sps* and *Cymbopogon martimii* showed lower conservation value standing at 28.7%, 39.2% and 42.8%, while the moisture content of soil was also lower i.e. 15.4%, 15.8% and 13.7%, respectively. On day 30 also the maximum soil conservation value was recorded in the case of *Cynodon dactylon* and *Adhatoda vasica* standing at 75.5% and 68.0%, the moisture content of soil was also higher being 20.7% and 21.4%, respectively

The experiment was conducted for a period of 45 days. Best results in respect of soil conservation value and soil moisture content were obtained with *Cynodon dactylon* and *Adhatoda vasica* which on day 45 stood respectively at

82.5%, 18.5% and 72.1% and 20.2%. *Ipomea carnea*, *Sida sps* and *Cymbopogon martimii* showed soil conservation value and soil moisture content which on day 45 stood at 36.2%, 14.6%, 12.2% and 48.4%, 15.0% respectively. The result shows that relatively high soil conservation value and soil moisture constent as obtained under the effect of *Cynodon dact ylon* and *Adhatoda vasica* are attributable to better higher leaf area index of these plants i.e. 0.89 and 0.98 cm^2, respectively. Besides, thir root system also was better developed which could tightly bind the soil particles against erosion. The higher leaf area of these plants provided a cover to the soil against the impact of rain drops so that the soil particles were relatively mildly hit resulting in the checking of soil erosion. The present findings are supported by earlier work where it was found that the dense grass cover checks the velocity of the rain drops and result is more water entering the soil and no erosion because their canopy and profuse root system tightly bind the soil against erosion, Donahue [4]. Grasses can be grown on lands which are not suitable for cultivation of crops. They are also used in protecting bunds, water ways, badly eroded areas etc. Kanitkar et al.[5].

In the light of the present study *Cynodon dact-ylon* and *Adhatoda vasica* are recommended for growing in the Shivalik for soil and moisture conservation. *Adhatoda vasica* is an important medicinal plant and hence its plantation in the Shivalik shall add to the medicinal plant wealth of the region that has been dwindling over the years as a consequence of over exploitation and ecodegradation in the region.

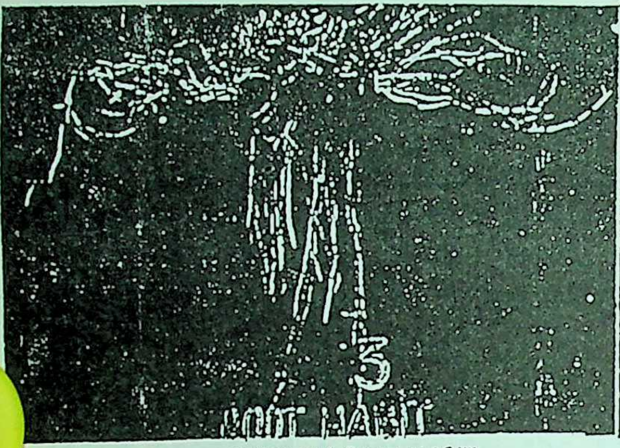
TABLE-1

MOISTURE CONTENT AND SOIL CONSERVATION VALUE OF DIFFERENT PLANTS

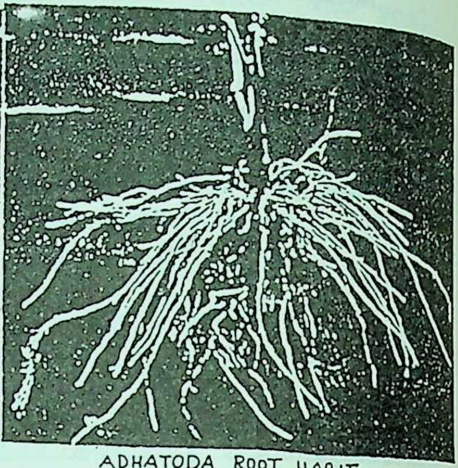
INTERVAL DAY	NAME OF PLANT SPS.	AIR T °C	SOIL T °C	HEIGHT OF PLANT (Cm)	LAI (Cm^2) (mean)	SOIL MOIS-TURE %	SCV *%	% OF NITROGEN
0	A	24.0	22	-	-	11.1	NIL	0.010
	B	24.0	21	12.6	0.06	21.9	45.0	0.012
	C	24.0	22	14.3	0.031	18.3	25.8	0.012
	D	24.0	21.5	13.2	0.52	19.6	49.2	0.013
	E	24.0	18	14.0	0.026	14.9	39.3	0.013
	F	24.0	16.5	10.8	0.05	16.2	37.6	0.012
15	A	24.8	23.6	-	-	10.8	NIL	0.010
	B	24.8	21.8	18.7	0.49	20.2	52.6	0.012
	C	24.8	23.0	16.8	0.08	15.4	28.7	0.011
	D	24.8	21.6	16.3	0.63	16.6	68.9	0.013
	E	24.8	19.6	18.9	0.032	13.7	42.8	0.013
	F	24.8	16.8	12.4	0.09	15.8	39.2	0.011
30	A	28.6	26.0	-	-	8.7	NIL	0.008
	B	28.6	24.2	29.8	0.86	21.4	68.0	0.012
	C	28.6	27.8	38.2	0.38	16.8	32.7	0.011
	D	28.6	24.0	27.5	0.72	20.7	76.5	0.013
	E	28.6	26.7	29.2	0.036	15.2	46.6	0.014
	F	28.6	25.4	26.8	0.42	14.4	41.2	0.012
45	A	29.7	26.9	-	-	6.0	NIL	0.006
	B	29.7	24.0	34.8	0.98	20.2	72.1	0.013
	C	29.7	29.8	39.8	0.66	14.6	36.2	0.012
	D	29.7	24.5	28.4	0.89	18.5	82.5	0.013
	E	29.7	26.4	30.8	0.47	15.0	48.4	0.014
	F	29.7	26.0	28.8	0.56	12.2	45.6	0.013

A = Control. B = *Adhatoda vasica* C = *Ipomea carnea* D = *Cynodon dacty*
E = *Cymobopogon martimii* . F = *Sida sps.*

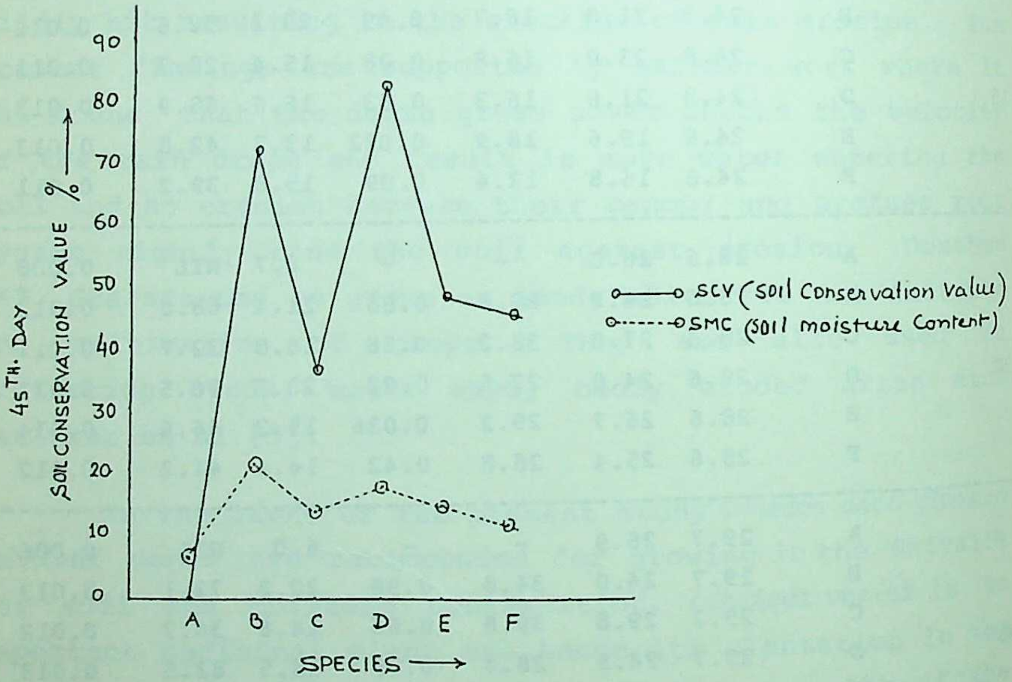
* Soil Conservation Value.



CYHODON ROOT HABIT



ADHATODA ROOT HABIT



(Fig 1)

A= control B= *Adhatoda vasica*, C= *Ipomea carnea*, D= *Cynodon dactylon*
E = *Cymbopogon martinii* F= *Sida sps.*

Fig 1 : Graphical representation of SCV and SMC by different plant species

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SIGNIFICANCE OF THE CONSTANT TERM OF A HERMITE POLYNOMIAL OF EVEN DEGREE

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ABSTRACT

We present new proofs of the definite parity of the Hermite polynomials and the recurrence relations satisfied by them. We also show that *all* the Hermite polynomials will be simple polynomials if the constant term of *any* Hermite polynomial of even degree is non-zero.

Keywords and Phrases : Hermite polymonials, Wronskian, Parity, Recurrence relations, constant term of a Hermite polynomial of even degree.

Mathematics Subject Classification Number:33A65

INTRODUCTION

The purpose of this paper is to present new proofs of the definite parity of the Hermite polynomials [1,2,4] and their recurrence relations and also to look at the significance of the constant term of *any* Hermite polynomial of even degree. If $H_{2n}(0)$ is non-zero for a *single* positive integer $n \geq 0$, the entire set of the Hermite polynomials, $[H_m(x)]$, will be a simple set of polynomials, each polynomial in the set having a non-zero leading term and hence a precise degree [4]. The differential equation, recurrence relations, and zeros of the Hermite polynomials are the windows through which we look at the beauty of $H_{2n}(0)$, the constant term of $H_{2n}(x)$.

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WRONSKIAN, PARITY AND RECURRENCE RELATIONS

The basis for our analysis is the differential equation satisfied by the Hermite polynomials [1,2,4]:

$$(D^2 - 2\,x\,D + 2p)\,H_p(x) = 0, \quad p = 0,1,2,\ldots, \tag{1}$$

where $$D \equiv d/dx. \tag{2}$$

The Wronskian of the two solutions Y_1 and Y_2 of the Hermite differential equation (HDE), $W_{HDE}(x)$, obtained using Abel's formula [3,5,6], is given by

$$W_{HDE}(x) = Y_1Y_2' - Y_2Y_1' = W_{HDE}(0)\exp(x^2). \tag{3}$$

Since $W_{HDE}(x)$ is an *even* function of x and also an *infinite* series, the two linearly independent solutions of the HDE cannot be polynomials simultaneously. (The Wronskian of two polynomial solutions will be another polynomial, leading to $W_{HDE}(0) = 0$.) Moreover, the two linearly independent solutions of the HDE cannot have the same parity. (The Wronskian will be an odd function in this case, yielding $W_{HDE}(0) = 0$.) As (1) is a homogeneous second order linear differential equation, it has $H_p(x) \equiv 0$ as a trivial solution. But we are interested in finding a non-trivial polynomial solution satisfying (1). When $x \rightarrow -x$, $D \rightarrow -D$, $D^2 \rightarrow D^2$, and $-2xD \rightarrow -2xD$. Hence if $H_p(x)$ is a non-trivial polynomial solution of the HDE, $H_p(-x)$ is a also solution of the same HDE. Since both $H_p(x)$ and $H_p(-x)$ are non-trivial polynomials, they are simply proportional to each other and hence $H_p(x)$ has a definite parity $(-1)^p$:

$$H_p(-x) = (-1)^p H_p(x). \qquad (4)$$

We next obtain, from the HDE, the differential recurrence relation satisfied by the Hermite polynomials. From (1), it is clear that $D(D^2-2xD+2p)\, H_p(x) = D\,0 = 0$. Hence we have.

$$[D^2 - 2xD + 2(p-1)]\,[DH_p(x) = 0, p \geq 1. \qquad (5)$$

Thus both $H_{p-1}(x)$ and $DH_p(x)$, $p \geq 1$, are solutions of the same HDE. Since both are non-trivial polynomials (they have the same parity too), they cannot be the linearly independent solutions of the HDE. Hence $DH_p(x) = C_p\, H_{p-1}(x)$, $p \geq 1$, C_p being the constant of proportionality. The familiar choice is $C_p = 2p$, for all $p \geq 1$. (The leading coefficient of *any* $H_p(x)$ cannot be obtained from the HDE. The above choice makes the leading coefficient of $H_p(x)$ to be 2^p.) Thus the differential recurrence relation is [1,2,4]:

$$DH_p(x) = 2p\, H_{p-1}(x), \; p \geq 1. \qquad (6)$$

From (1), we have $[D^2 - 2\,x\,D + 2(N+1)]\, H_{N+1}(x) = 0$. Eliminating the derivatives using (6), we arrive at the three-term recurrence relation satisfied by the Hermite polynomials [1,2,4] :

$$H_{N+1}(x) = 2\,x\,H_N(x) - 2\,N\,H_{N-1}(x),\ N \geq 1. \tag{7}$$

Using (4), we have $H_{2m+1}(0) = 0$. One may be tempted to ask whether $H_{2m}(0)$ can also follow suit. We show that the answer is definitely in the negative.

SIGNIFICANCE OF $H_{2m}(0) \neq 0$ AS SEEN THROUGH THE WINDOW OF THE HDE

The HDE for even degree polynomials is

$$(D^2 - 2\,x\,D + 4m)\,H_{2m}(x) = 0,\ m \geq 0. \tag{8}$$

Since $H_{2m}(x)$ is an even polynomial in x (see (4)), its derivative with respect to x is an odd polynomial in x and hence vanishes at x = 0. If the constant term of $H_{2m}(x)$ were zero, then the solution of (8) and its derivative would be zero at the same point x = 0. As an initial value problem in the theory of second order linear differential equations has a unique solution [5,6], the solution of (8) would be identically zero. As we are interested in having a non-trivial polynomial solution of the HDE, we must choose $H_{2m}(0)$ to be non-zero.

What we give below is a "constructive" proof of the fact that $H_{2m}(x)$ is non-trivial, with a non-zero leading term. This is an alternative to the one generally given in books [1,6] for finding the polynomial solution from the HDE. Differentiating both sides of (8) with respect to x successively 2k times, using the Leibnitz theorem, we have

$$D^{2k+2}H_{2m}(x)|_{x=0} = 4(k - m)\,D^{2k}\,H_{2m}(x)|_{x=0}. \tag{9}$$

As $H_{2m}(0)$ is non-zero, $m \geq 1$, iteration of (9) shows that $D^{2s}\ H_{2m}(x)$ has to be non-zero at $x = 0$ when $0 \leq s \leq m$. If

$$H_{2m}(x) = \sum_{s=0}^{m} A_{2s}\ x^{2s}, \tag{10}$$

then

$$A_{2s} = (-1)^s\ 4^s\ m!\ H_{2m}(0)/[(2s)!\ (m-s)!] \neq 0. \tag{11}$$

Using (1), (4) and (6), we now show that $H_{2m}(0) \neq 0$ will also make $H_{2m+1}(x)$ a non-trivial polynomial. The Hermite polynomial of odd degree is an odd polynomial and hence is zero at the origin. The differential equation satisfied by it implies that it is not identically zero since its derivative is different from zero at the origin: $DH_{2m+1}(x)|_{x=0} = 2(2m+1)H_{2m}(0) \neq 0$.

It follows from the differential recurrence relation and (8) that

$$\begin{aligned} 4(2m+2)\ (2m+1)\ H_{2m}(0) &= D^2\ H_{2m+2}(x)|_{x=0} \\ &= -4\ (m+1)\ H_{2m+2}(0). \end{aligned} \tag{12}$$

Hence $H_{2m+2}(0) \neq 0$, if $H_{2m}(0)$ is non-zero for a particular m. Iteration of (12) now yields that $H_{2m}(0) \neq 0$, for all $m \geq 0$. Since every non-zero $H_{2m}(0)$ saves both $H_{2m}(x)$ and $H_{2m+1}(x)$ from being identically zero, it is clear that a single non-zero $H_{2m}(0)$ can save the entire set of Hermite polynomials.

THROUGH THE THREE-TERM RECURRENCE RELATION AND ZEROS

From the three-term recurrence relation, (7), we have

$$H_{2m}(0) = -2(2m-1)H_{2m-2}(0), \; m \geq 1 . \tag{13}$$

As every polynomial belonging to a set of simple polynomials has a precise degree with a non-zero leading term [4,11], $H_0(0)$ cannot be zero if we demand the Hermite polynomials to be simple polynomials. From (13) it follows that $H_{2m}(0) \neq 0$, $m \geq 0$.

If the constant term of $H_{2r}(x)$ were zero for some $r \geq 1$, then $H_{2r}(x) = x^2 [A_{2r} x^{2r-2} + \ldots + A_2]$. (Here we use only the fact that $H_{2r}(x)$ has a definite parity (see (4)) and hence it contains only even powers of x.) Therefore, $H_{2r}(x)$ would have a zero of multiplicity 2 at $x = 0$, contradicting the fact that the zeros of real orthogonal polynomials are distinct [4,11]. (This result is a consequence of the orthogonal property [4,11] which itself is a natural consequence of the Sturm-Liouville form of the HDE [1,3,5]. See Section 6.) Hence $H_{2r}(0) \neq 0$, for all $r \geq 0$.

OTHER WINDOWS

Readers may wonder why we have not looked either through the window of the generating function [1,2,4]

$$\exp(2xt - t^2) = \sum_{m=0}^{\infty} H_m(x)\, t^m/m!, \tag{14}$$

or through that of the Rodrigues' representation [1,2,4,

$$H_n(x) = (-1)^n \exp(x^2)\, D^n(\exp(-x^2)),\ n \geq 0, \qquad (15)$$

or through that of the simple but beautiful relation [1,7,8,9]

$$H_n(x) = (2x - D)^n\, 1,\ \ n \geq 0. \qquad (16)$$

Each one of the above three relations has been proved to be a starting point for studying the properties of the Hermite polynomials. For proofs, see Refs. [2], [7], and [8,9], respectively. Like the power series expansion [1,2,4] of $H_n(x)$, they specify the Hermite polynomials completely, leaving no room for any speculation whether $H_{2n}(0)$, $n \geq 0$, can be zero or not.

STRUM-LIOUVILLE FORM AND SZEGÖ'S RESULT

The Strum-Liouville form of the HDE is [1,3,5]

$$[L + 2p \exp(-x^2)]\, H_p(x) = 0, \quad p \geq 0, \qquad (17)$$

where the Strum-Liouville operator L is $D \exp(-x^2) D$. As the eigenfunctions of the Sturm-Liouville operator belonging to distinct eigenvalues are orthogonal with respect to the weight function [1,3,5],

$$\int_{-\infty}^{+\infty} \exp(-x^2)\, H_m(x)\, H_n(x)\, dx = 0, \quad m \neq n. \qquad (18)$$

The even nature of the associated weight function and the symmetry of the interval of orthogonality with respect to the origin demand, according to an elegent and general result from Szegö [11] that $H_n(x)$ *can contain* only those powers of x which are congruent to n(mod 2), i.e., it is an odd or even polynomial according to whether n is odd or even. Using Szegö's result [11], the properties of the zeros [4,11] of $H_n(x)$ and Descartes' rule of signs, it has been shown [10] that $H_n(x)$ *does contain* only those and all those powers of x which are congruent to n(mod 2). Hence in

$$H_{2n}(x) = \sum_{m=0}^{n} A_{2m} x^{2m}, \ n \geq 0, \tag{19}$$

$$H_{2n+1}(x) = x \sum_{s=0}^{n} A_{2s+1} x^{2s}, \ n \geq 0, \tag{20}$$

of the coefficients A_{2m} and A_{2s+1} can be zero. As simple corollaries, we have,

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) \neq 0, \quad n \geq 0 \tag{21}$$

With $H_0(0) = 1$, it follows from (12) or (13) and iteration that

$$H_{2n}(0) = (-1)^n \, (2n)!/n! \neq 0, \ \ n \geq 0. \tag{22}$$

CONCLUSIONS

What a tremendous potential the particularly pleasant property $H_{2n}(0) \neq 0$ has! As $H_{2n}(0)$ 0, $H_N(x) \not\equiv 0$, $N \geq 0$. One member of a family saves the entire family: The constant term of a Hermite polynomial of even degree is the saviour of the Hermite polynomials. Our analysis throws light on the usefulness of the Wronskian in obtaining the parity and recurrence relations satisfied by the Hermite polynomials. While it is good to know a better proof of a result, it is better to know good proofs of the same also. One may extend our approach to other real orthogonal polynomials associated with an even weight function and an interval of orthogonality symmetric with respect to the origin (e.g. Legendre polynomials and Chebyshev polynomials [1,2]).

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A NOTE ON THE MODULAR GROUP RING OF THE SYMMETRIC GROUP S_n

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Keyword : The mod p-envelp of a group.

Mathematics Subject Classification : 16A26 & 20E30.

In [2] the author has proved if G is a group of order two and K is a field with more than two elements then G^*, the mod p-envelope of G has a semi-group structure with a non trivial idempotent in it. It is interesting to note when G is the symmetric group S_n and K is a field having two elements 0 and 1 then G^* the mod p-envelope of G has a semi-group structure with non trivial idempotents in it and the number of elements in G^* is $2^{\lfloor n-1}$. For the sake of completeness we prove the following propositions. For definition's please refer [1] and [2].

Proposition 1. Let G be the symmetric group of permutations of (1,2,3) and K = (0,1) be the field of two elements. Then G^* is a semi-group with non trivial idempotents and the number of elements in G^* is $32 = 2^5 = 2^{\lfloor 3-1}$.

Proof. $G = \{ \begin{pmatrix}1\,2\,3\\1\,2\,3\end{pmatrix} = 1, \begin{pmatrix}1\,2\,3\\1\,3\,2\end{pmatrix} = p_1, \begin{pmatrix}1\,2\,3\\3\,2\,1\end{pmatrix} = p_2, \begin{pmatrix}1\,2\,3\\2\,1\,3\end{pmatrix} = p_3,$

$\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix} = p_4, \begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix} = p_5 \}.$

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$$\begin{aligned} G^* &= \{1+U\} \\ &= \{1, p_1, p_2, p_3, p_4, p_5, 1+p_1+p_2, 1+p_2+p_3, 1+p_1+p_3, \\ &\quad 1+p_1+p_4, 1+p_1+p_5, 1+p_2+p_4, 1+p_2+p_5, 1+p_3+p_4, \\ &\quad 1+p_3+p_5, 1+p_4+p_5, p_1+p_2+p_3, p_1+p_2+p_4, p_1+p_2+p_5, \\ &\quad p_1+p_3+p_4, p_1+p_3+p_5, p_1+p_4+p_5, p_2+p_4+p_5, p_3+p_4+p_5 \\ &\quad p_2+p_3+p_4, p_2+p_3+p_5, 1+p_1+p_2+p_3+p_4, 1+p_1+p_2+p_3+p_5, \\ &\quad 1+p_1+p_2+p_4+p_5, 1+p_2+p_3+p_4+p_5, 1+p_1+p_3+p_4+p_5, \\ &\quad p_1+p_2+p_4+p_5+p_3\}. \end{aligned}$$

Clearly G^* is a semi-group with identity

$$\begin{aligned} &\{1+p_4+p_5, p_1+p_2+p_4, p_1+p_2+p_5, p_1+p_3+p_4, p_1+p_3+p_5, \\ &\quad p_2+p_3+p_3, p_2+p_3+p_5, 1+p_1+p_2+p_4+p_5, 1+p_1+p_3+p_4+p_5 \\ &\quad \text{and } p_1+p_2+p_3+p_4+p_5\} \end{aligned}$$

are nontrivial idempotents of G^*.

Proposition 2. Let G be the symmetric group of permutations of $(1,2,3,4)$ and $K=(0,1)$ be the field of two elements. Then G^* is a semi-group with non trivial idempotents and the number elements in G^* is $2^{23} = 2^{\underline{|4}-1}$.

Proof. $G^* = \{1+U\}$

$$= \{\alpha = \Sigma\alpha_i g_i \,/\, \Sigma\alpha_i = 1 \quad g_i \in G,\ \alpha_i \in (0,1)\}$$

$$|G^*| = 2[24C_1+24C_3+24C_5+24C_7+24C_9+24C_{11}]$$

$= 2^{23}$ (I have verified by calculations).

Consider

$$\alpha_1 = 1 + \begin{pmatrix} 1&2&3&4 \\ 3&2&4&1 \end{pmatrix} + \begin{pmatrix} 1&2&3&4 \\ 4&2&1&3 \end{pmatrix}$$

$$\alpha_2 = 1 + \begin{pmatrix} 1&2&3&4 \\ 2&3&1&4 \end{pmatrix} + \begin{pmatrix} 1&2&3&4 \\ 3&1&2&4 \end{pmatrix} \text{ and}$$

$$\alpha_3 = 1 + \begin{pmatrix} 1&2&3&4 \\ 2&4&3&1 \end{pmatrix} + \begin{pmatrix} 1&2&3&4 \\ 4&1&3&2 \end{pmatrix}$$

are some of the non-trivial idempotents in G^*.

Promblem. Derive a method by which we can enumerate the number of non-trivial idempotents in G^*.

Theorem 3. Let G be the symmetric group of degree n. Let $K = (0,1)$ be the field of two elements, G^* the mod p envelope of G has $2^{\lfloor n-1}$ elements and it has non-trivial idempotents and forms a semigroup.

Proof.

$$\begin{aligned} G^* &= \{1 + U\} \\ &= \lfloor nC_1 + \lfloor nC_3 + \lfloor nC_5 + \ldots + \lfloor nC_{\lfloor n-1} \\ &= 2^{\lfloor n-1}. \end{aligned}$$

Now every element in G^* is such that the sum of the coefficients is 1. This forces the number of terms of every element in G^* is odd. Clearly G^* is closed for the product of odd terms modulo two is also odd. Hence G^* is semi-group. $1 \in G^*$ so G^* is also a semi-group with identity. G^* has non-trivial idempotents for take.

$$\alpha = 1 + \begin{pmatrix} 1&2&3&4&\ldots&n \\ 2&3&1&4&\ldots&n \end{pmatrix} + \begin{pmatrix} 1&2&3&4&\ldots&n \\ 3&1&2&4&\ldots&n \end{pmatrix}.$$

Clearly $\alpha^2 = \alpha$ hence $\alpha \in G^*$ is a non trivial idempotent.

Example 1. Let $S_2 = \{ \begin{pmatrix} 1&2 \\ 1&2 \end{pmatrix}, \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \}$, $K = (0,1,2)$ be the field of three elements. What is G^*?

Proof. $G^* = \{1+U\}$

$$= \{1, \begin{pmatrix}1&2\\2&1\end{pmatrix}, 2+2\begin{pmatrix}1&2\\2&1\end{pmatrix}\}.$$

Order of $G^* = 3^{\underline{|2}-1} = 3$ and $2+2\begin{pmatrix}1&2\\2&1\end{pmatrix}$ is the non-trivial idempotent of G^*.

Example 1 can be extended to any field of p-elements

p a prime $G = \{\begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&1\end{pmatrix}\}$ and let

$$K = (0,1,2,\ldots,p-1)$$

$$G^* = \{1, (p-1)\begin{pmatrix}1&2\\2&1\end{pmatrix}+2, 2\begin{pmatrix}1&2\\2&1\end{pmatrix}+p-1, (p-2)\begin{pmatrix}1&2\\2&1\end{pmatrix}+3, 3\begin{pmatrix}1&2\\2&1\end{pmatrix}+p-2, \ldots, \frac{p+1}{2}+\frac{p+1}{2}\begin{pmatrix}1&2\\2&1\end{pmatrix}\}.$$

G^* is a semi-group with p-elements, and $\frac{p+1}{2}+\frac{p+1}{2}\begin{pmatrix}1&2\\2&1\end{pmatrix}$ is an idempotent of G^*. Then order of

$$G^* = p^{\underline{|2}-1} = p.$$

Hence we generalize this as a problem.

Problem. If G is symmetric group of order n, $K = (0,1,2,\ldots,p-1)$ be a semi-group of order $p^{\underline{|n}-1}$ and has non-trivial idempotents.

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AN ALTERNATIVE APPROACH TO TOLLMIEN'S ENERGY SPECTRA DERIVED FROM HEISENBERG'S EQUATION OF TURBULENT FLOW

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ABSTRACT

Tollmien (1952-53) had to work with experimenental data of windtunnel measurements due to Stewart and Townsend (1951) for the evaluation of turbulent energy spectra. In this paper similar categories of spectra have been shown to be alternatively derivable from Sen's (1951) self preserving solution of Heisenberg's (1948) equation and by extending numerical tables of Ghosh (1955) the author has ultimately obtained self-similar spectrum curves somewhat analogous to the normalized apectra calculated by Tollmien at different times of turbuleNt fluid flow.

1. INTRODUCTION

It is well known that experimental studies o turbulence are based on relevant data obtaine experimentally at least at a distance of 20 mesh lengt behind the grids of wind tunnel. Tollmien's numerica evaluation of turbulent energy spectra (1952-53)at times t= t=0.2, t=0.5,t=1 has a footing on experimental results a 30mesh length. If F(K,t) be the energy spectrum function o homogeneous- isotropic turbulence for wave number K at tim t,the decay of turbulence occurs according to Heisenberg' (1948) equation

$$-\frac{\partial}{\partial t}\int_{\infty}^{k} F(k,t)dk=2(\nu+n_k)\int_{\infty}^{k} k^2F(k,t)dk \qquad \ldots (1$$

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where ν is kinematic viscosity and n_k is the turbulent viscosity given by

$$n_k = \nu_H \int_k^{\infty} \frac{d\, F(K,t)}{k^3}\, dk$$

ν_H being a constant.

Since the integro-differential equation (1) contains only the first order time derivative, is follows that its solution is uniquely determined by imposing initial condition of the form

$$F(K,t)\big|_{t\to 0} = F(K,0) = F_0(K), \text{ (say).}$$

Therefore, if the form of the spectrum at initial time is known from experiment, one can determine the spectrum for all subsequent times by numerical integration Thist type of calculation has been incidentally carried out by Tollmien for Heisenberg's equation (1) and the data of Stewart and Townsend(1951). Tollmien took $F_0(K)$ in the form of data of Fig.(a) and Y_H=0.45 according to Proudman's estimate (1951). Furthermore, Tollmien attempted to normalize his solution in accordance with the data of Fig.(a) so that the values of F(K,t) for large K were as near time-independent as possible. The curves obtained in this way vide Fig.(b) were found to be qualitattively similar to empirical curves of Stewart and Townsend shown in Fig. (a). Although it has been stated that the resulting function F(K,t) did not display self preserving behaviour in time, yet the current evaluation will lead to inference otherwise if we start with Ghosh's (1955) numerical tables.

Nevertheless, turbulent energy calculated by Tollmien was found to be in very good agreement with the empirical "minus one law" right upto t=0.5 and this law was usually regarded as an evidence for self-preservation.

In tins context it may be mentioned that Karman-Howarth(1938) defined self preserving solutions as to be those whose curves do not undergo a change of shape with time, the only change being the change of scale. The motion in a sense remains similar to itself.

Heisenberg has shown that the equation (1) admits of a self similar solution of the form for

$$F(k,t) = \frac{1}{\sqrt{t}} f(K\sqrt{t}) \qquad \ldots (2)$$

for which f(x) with x=kt obeys an ordinary differential equation derivable from (1). This set of self-similar solutions was numerically calculated by Chandrashekhar (1949). Such numerical solutions so obtained on the basis of equation (2) have obvious break of similarity at the left end of self similar curves [cf. Karman & Lin (1951)].

In order to overcome this drawbhack Sen(1951) suggested a bunch of solutions of Heisenberg's equation (1) in the form

$$F(K,t) = \frac{S^3}{\nu^2 k_o^3 t_o^3 \tau^2} f\left(S\frac{K}{K_o}\right) \qquad \ldots (3)$$

where $s=s(\tau)$, $T=t/t_o$ and K_o, t_o are constant. Using Reynolds number R as

$$\frac{1}{R} = \nu k_o^2 t_o \text{ and } SK/K_o = x,$$

the equation of f(x) is obtined by substitution of (3) in (1 as

$$2(1-c)\int_o^x f(x)dx - cxf(x) = 2\int_x^\infty \sqrt{\frac{f(x)}{x^3}}\,dx \int_o^x x^2 f(x)dx + \frac{2\tau}{S^2R} \qquad \ldots (4)$$

where c is parameter introduced by Sen as $c = \frac{1}{s}\frac{ds}{dt}$ i.e. $s = \alpha\tau^c$, α being a constant.

Neglecting the last term of equation (4) for early period of decay process where Reynolds number of fluid flow is infinitely large, Ghosh has obtained numerical tables by means of Sen's self similar solutions.

$$F(k,t) = \left[\frac{1}{\nu_H^2 k_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{3c-2} f\left(\frac{k}{k_o} \left\{\frac{t}{t_o}\right\}\right)\right]_{c=\frac{2}{7}}$$

with asymtotic behaviours

$$f(x) \approx x^4, (x \to 0) \text{ and } f(x) \sim x^{-5/3}, (x \to \alpha).$$

In doing so Ghosh also followed Chandrasekhar for introduction of two auxiliary functions as

$$g = x^3 f(x) \text{ and } y(x) = \int_o^x x^2 f(x)\,dx$$

to recast the equation (4) for $R \to \alpha$ into the form

$$cg^{3/2}g'' + (4+g')y + 2g^{1/2}(2-cg') - cg) - 4g = 0 \qquad \ldots (5)$$

where $g' = \frac{dg}{dy}$.

If $c=\frac{1}{2}$, the relation (5) gives Chandrasekhar's equation for the preparation of numerical table of (y,g(y)) and thereafter (x,f(x)), whereas $c = \frac{2}{7}$ leads to Ghosh's tables for (y,g(y)], and consequently (x,f(x)). It is observed that desipite (x,f(x)) curves of Chandrasekhar's tabular values and those due to Ghosh are by and large similar in pattern, yet it is obvious that there is a break of similarity at the left end portion of each individual (x,f(x)) curves due to Chandrasekhar's tables. This is because of the fact that the asymptotic behaviours of Chandrasekhar's tables involve $f(x) \approx x$, $(x \to o)$ and $f(x) \approx x^{-5/3}$, $(x \to \alpha)$ of which, the first one i.e. $f(x) \approx x, (x \to 0)$ leads to break of similarity while $F(K,t)|_{t \to o}$ is to be accounted for through a behaviour of f(x) at $x \to 0$. Hence to proceed with this form for $f(x) = f\{\frac{k}{k_o} \sqrt{\frac{t}{t_o}}\}$ as $t \to 0$ is anomalous.

As such in the present paper it has been attempted to generate numerical values of F(K,t) when $t \to 0$ as obtainable from Sen's self-similar solution of equation (1) for parametric value $c = \frac{2}{7}$ by making an extension of Ghosh's numerical results transformed into new tables that eventually yield identical curves resembling Tollmien's spectra and by dint of this approach it became possible to maintain the self preserving behaviour throughout the entire specturm as well.

2. APPROACH TO THE POBLEM

In the present problem the equation (1) has, in fine, the more general solution of the from

$$F(k,t) \approx \text{const.}\, t^{3c-2} f(kt^c)$$

where c is an arbitarary constant.

For $c = \frac{2}{7}$, $(0 < c < \frac{2}{3})$ we have in particular

$$F(k,t) \approx \frac{1}{\nu_H^2 K_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{-8/7} f\left(\frac{k}{k_o}\left\{\frac{t}{t_o}\right\}^{2/7}\right) \qquad \dots (6)$$

Evidently asymptotic behaviours of F(k,t) are here found to be

$$F(k,t) \approx k^4, (k \to 0) \quad F(k,t) = k^{-5/3}, (k \to \infty)$$

The first one of the asymptotes was supported in the works of C.C. Lim (1947) and G.K. Batchelor (1949), while the other asymptotic behaviour leads to Kolmogorov (1941) spectrum. In addition to theoretical background, in fact the spectrum K^4, $(k \to 0)$ has got also strong experimental confirmation according to Debnath (1978) and the evaluation of F(k,t) for $c = \frac{2}{7}$ concerns in this case the derivation of numerical solutions of (x,f(x)) without any break of self-similarity at $x \to o$ i.e. $t \to o$ or $k \to o$ as well as $t \to o$. According to our requirement we are to set forth the intial conditions as to be

$$F(k,t)\big|_{t \to o} = \frac{1}{\nu_H^2 K_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{-8/7} [f(x)]_{x \to o} \text{ for } x = \frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7} \qquad \ldots (7)$$

where $(x, f(x))$ data are to be collected from numerical tables of Ghosh and K_o, t_o being unity for the present instance. The reliability of new set of results may be verified by the condition $F(k,t) = \Lambda k^4$ as $k \to o$, where Λ is Loitsiansky's (1939) invariant in respect of the tables concerned

3. NUMERICAL TABLES AND GRAPHS

In the light of aforesaid facts we have developed the following numerical tables I & II where first, second columns represent data taken from Ghosh's tables and for $x = \frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7}$ the numerical values of third, fourth, fifth columns of each table have been calculated at t=0.5, 0.2, 0.1 respectively. On the basis of such extended tables we have generated turbulent energy spectrum converted finally in terms of curves (Fig.1 and Fig.2) which are found to be akin to the Tollmien's spectra and thereby leading to remarkable modification in the self preserving nature of spectra as pictured by the current approach.

CONCLUDING REMARKS

(i) Thus to start with fourth power law $f(x) \approx x^4, (x \to 0)$ meaning $x = \frac{k}{k_o} (\frac{t}{t_o})^{2/7} \to 0$ includes the behaviour at $t \to 0$ and hence having regard to Sen's (1951) self similar solution of Heisenberg's (1948) energy decay equation one can safely work with the tables of Ghosh (1955) in lieu of wind-tunnel data of Stewart and Townsend (1951) for the evaluation of turbulent energy spectrum at different times where the resulting function F(k,t) exhibits self preserving behaviour with time as evident from the graph of the present note.

(ii) One may extrapolate freely at the end of the tables extended in the current work as there is no break of self similarity anywhere in the whole range of the spectra and such extrapolation at the extreme upper end of the tables concerns the evaluation of Loitsianky's constant Λ that remains independent of time.

(iii) At the initial stage of decay phenomean when low frequency part of the spectrum plays significant role or in otherwords when the low frequency eddies are predominant, then variety of spectra satisfying similarity law may be possible depending on the method of production of turbulence as the parameter c being proposed to be associated with the mode of excitation of turbulence and this approach is valid at a stage of turbulence of negligible viscous dissipation by the wave numbers concerned so that the inertia terms of Navier-stokes equation are mainly responsible for the entire transfer of energy down the spectrum of such self preserving isotropic turbulence.

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TABLE : 1

x	f(x)	k	k	k
0.3737	0.1314	0.4555	0.5919	0.7215
0.4166	0.1660	0.5078	0.6598	0.8043
0.4708	0.2067	0.5739	0.7456	0.9090
0.4910	0.2203	0.5985	0.7777	0.9480
0.5320	0.2442	06485	0.8426	1.0271
0.5656	0.2592	0.6895	0.8958	1.0920
0.6623	0.2792	0.8074	1.0490	1.2787
0.7360	0.2720	0.8972	1.1657	1.4210
0.8006	0.2525	0.9759	1.2680	1.5457
0.8607	0.2279	1.0492	1.3632	1.6618
0.9200	0.1979	1.1215	1.4571	1.7762
0.9810	0.1664	1.1959	1.5537	1.8940
1.0466	0.1355	1.2758	1.6576	2.0207
1.1206	0.1003	1.3660	1.7748	2.1635
1.2111	0.0668	1.4763	1.9182	2.3383
1.8446	0.0037	2.2486	2.9215	3.5614

TABLE : II

x	f(x)	k	k	k
0.3736	0.1320	0.4554	0.5917	0.7213
0.4160	0.1695	0.5071	0.6589	0.8032
0.4687	0.2158	0.5714	0.7423	0.9049
0.4882	0.2323	0.5951	0.7732	0.9426
0.5273	0.2628	0.6428	0.8351	1.0181
0.5582	0.2845	0.6805	0.8841	1.0777
0.6469	0.3295	0.7886	1.0246	1.2490
0.7109	0.3452	0.8666	1.1259	1.3725
0.7638	0.3481	0.9311	1.2097	1.4747
0.8104	0.3436	0.9879	1.2835	1.5646
0.8530	0.3346	1.0398	1.3510	1.6469
0.9310	0.3083	1.1349	1.4745	1.7975
1.0042	0.2757	1.2241	1.5905	1.9388
1.0759	0.2397	1.3115	1.7040	2.0772
1.1492	0.2019	1.4009	1.8201	2.2186
1.2270	0.1633	1.4957	1.9433	2.3690
1.3137	0.1245	1.6014	2.0807	2.5364
1.4168	0.0861	1.7271	2.2440	2.7354
1.5547	0.0490	1.8952	2.4624	3.0016
1.8078	0.0147	2.2037	2.8632	3.4903
2.3858	0.0009	2.9083	3.7787	4.6063

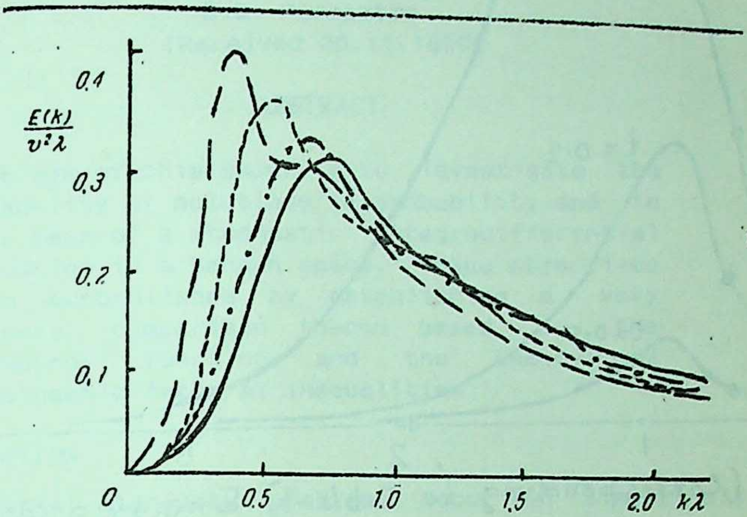

FIG. (a) Normalized spectral density functions at different distances from the grid [Stewart and Townsend (1951)].

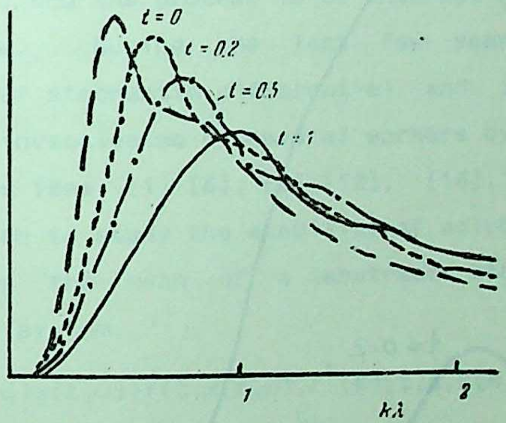

FIG. (b) Normalized spectra $F(k,t)$ for different t, calculated by Tollmien (1952 - 1953).

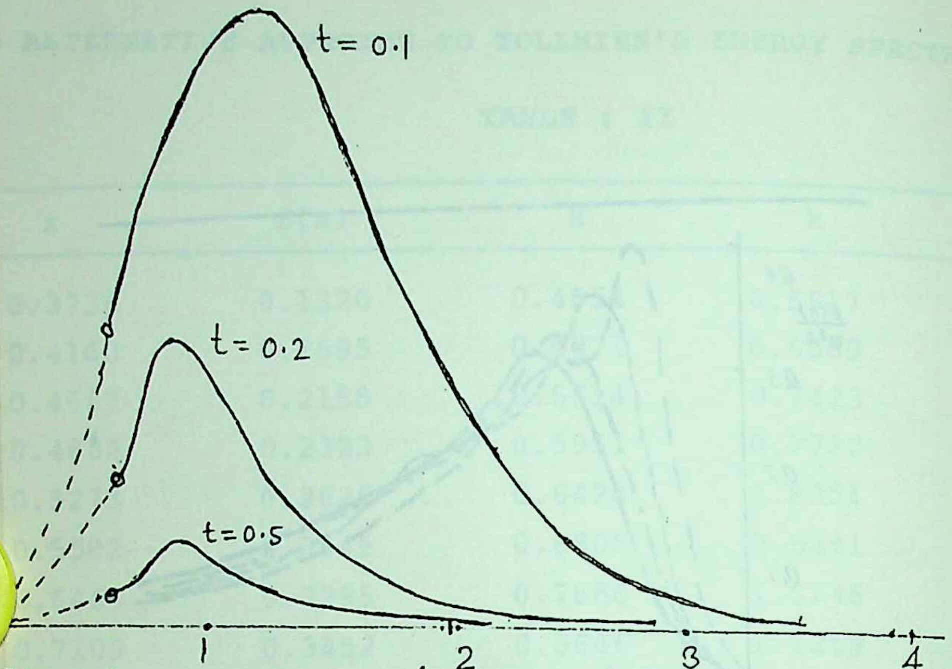

FIG.1 (Corresponding to Table-I) Energy decay spectra $F(k,t)$.

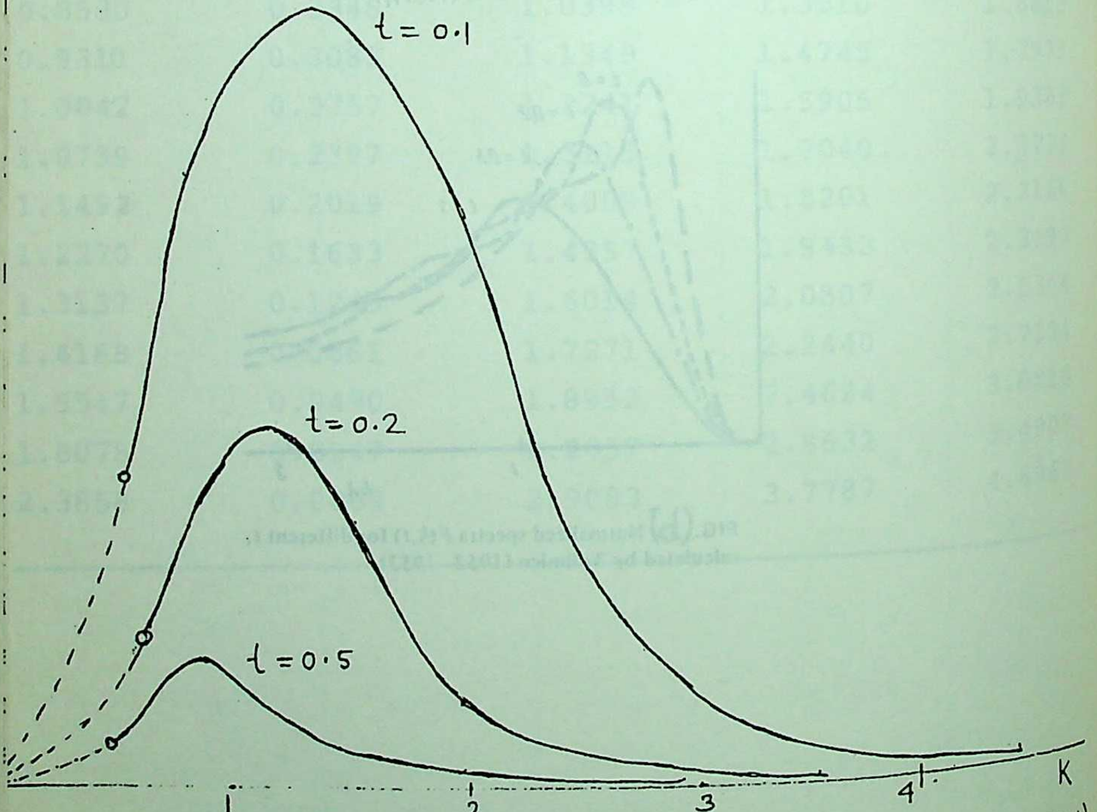

FIG.2 (Corresponding to Table-II) Energy decay spectra $F(k,t)$.

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ON THE STABILITY OF A STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS IN A BANACH SPACE

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ABSTRACT

The aim of this paper is to investigate the stability of solutions in probability and in the mean of a stochastic integrodifferential equation in a Banach space. These objectives are accomplished by establishing a very general comparison theorm based on the Lyapunov function and the theory of stochastic integral inequalities.

1. INTRODUCTION

Stochastic stability problems occur in almost all phases of physics, control theory, numerical analysis and economics where dynamical models subject to random disturbances appear, and the process is of interest over a long period of time. During the last few years, the stability analysis of stochastic differential and integral equations has been investigated by several workers by using different techniques (see, [1]-[5], [8]-[12], [14], [15]). In this paper we wish to study the stability of solutions in probability and in the mean of a abstract stochastic integrodifferential system.

$$(1)\quad x'(t,\omega) = A(t,\omega)x(t,\omega)+f(t,x(t,\omega),\int_{t_0}^{t}(K)[t,s,x(s,\omega),\omega]ds,\omega)$$

$$+g(t,x(t,\omega),\omega),\quad x(t_0,\omega) = x_0(\omega),$$

given certain connective information concerning the solutions of the corresponding abstract stochastic integrodifferential system

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(2) $$y'(t,\omega) = A(t,\omega)y(t,\omega)+f[t,y(t,\omega), \int_{t_0}^{t}K[t,s,y(s,\omega),\omega]ds,\omega],$$

$$y(t_0,\omega) = x_0(\omega),$$

and the auxiliary stochastic integrodifferential equation

(3) $$r'(t,\omega)= H_1(t,r(t,\omega),\int_{t_0}^{t}K_0[t,s,r(s,\omega),\omega]ds,\omega)+H_2(t,r(t,\omega),\omega),$$

$$r(t_0\omega) = r_0(\omega),$$

where

(i) $\omega \in \Omega$ the supporting set of a complete probability measure space (Ω,F,P);

(ii) $x(t,\omega)$, $y(t,\omega)$ are the unknown random processes which map $R^+ \times \Omega$ into X, where X is a seperable Banach space with norm $\| \,.\, \|$;

(iii) for each $t \in R^+$, the set $A(t,\omega)$ is a family of random operators [1] in X, with domain $D[A(t,\omega)]$;

(iv) $g(t,x,\omega)$, $K[t,s,x,\omega]$ and $f(t,x,\bar{x},\omega)$ are Borel measurable maps from $R^+ \times X \times \Omega$, $R^+ \times R^+ \times X \times \Omega$ and $R^+ \times X \times X \times \Omega$ into X respectively such that $g(t,0,\omega) = 0$, $f(t,0, \int_{t_0}^{t}(K)[t,s,0,\omega]ds,\omega) = 0$;

(v) $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ and $H_1(t,r,\bar{r},\omega)$ are Borel measurable maps from $R^+ \times R \times \Omega$, $R^+ \times R^+ R \times \Omega$ and $R^+ \times R \times R \times \Omega$ into R respectively such that $H_2(t,0,\omega) = 0$, $H_1(t,0,\int K_0[t,s,0,\omega], \omega) = 0$.

Recently, by assuming the existence of solutions of stochastic differential and integral equations Kats and Krasovskii [2], Martinjuk [8], Ladde [4], Ladde, Lakshmikantham and Liu [5], Rao and Manougian [14], Pachpatte [10]-[12] and others have studied the stability and boundedness of solutions by employing the second method of Lyapunov and the theory of differential and integral inequalities. Here, by assuming the existence of global mean square continuous solutions of (1), (2) and (3) for all $t \geq t_0$ and employing Lyapunov functions and the theory of integral inequalities we give sufficient conditions for the stability in the probability and in the mean of the solutions of (1) by means of comparison with the connective stablility in the probability and in the mean of (2) and (3).

2. PRELIMINARIES

In our subsequent discussion, we are interested in random linear operators difined on subsets of $X \times \Omega$. Suppose that for each $t \in R^+$, $A(t,\omega)$ is a closed random linear operator such that $[I-\alpha A(t,\omega)]^{-1}$ exists for a real α, where I is the identity operator. Suppose further that $D[I-\alpha A(t,\omega)]^{-1}$ is dense in X. Denote the resolvant of $A(t,\omega)$ by $R(\alpha, A(t,\omega)) \equiv [I\alpha A(t,\omega)]^{-1}$. Then the following results are known [1]:

(i) $R(\alpha,A(t,\omega))$ is a random linear operator for each $t \in R^+$;

(ii) $D[R(\alpha,A(t,\omega))] = X$;

(iii) $[I-\alpha A(t,\omega)]\ R(\alpha,A(t,\omega))x = x$ for $x \in X$;

(iv) $R(\alpha,A(t,\omega))\ [I-\alpha A(t,\omega)]x = x$ for $x \in D\ [A(t,\omega)]$.

We shall use the following definitions in our subsequent discussion, (see,[4, 5, 9, 12]).

Definition 1. Let $x(t,\omega) \in D[A(t,\omega)]$ be a random variable with values in X for each $t \in R^+$. Then $x(t,\omega)$ is called a solution of (1) if $x(t,\omega)$ has a strong derivative in X and satisfies (1).

Definition 2. The trivial solution of (1) is said to be (SP_1) *Stable in probability*, if for each $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0, \varepsilon, \eta) > 0$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$$

implies

$$P\{\omega : \|x(t,\omega)\| \geq \varepsilon\} < \eta, \; t \geq t_0;$$

(SP_2) *asymptotically stable in probability*, if it is stable in probability and, if for any $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exist positive number $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \varepsilon\eta)$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$$

implies

$$P\{\omega : \|x(t,\omega)\| \geq \varepsilon\} < \eta, \; t \geq t_0 + T;$$

(SM_1) *Stable in the mean*, if for each $\varepsilon > 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0, \in) > 0$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta$$

implies

$$E[\|x(t,\omega)\|] < \varepsilon, \; t \geq t_0;$$

(SM_2) *asymptotically stable in probability*, if it is stable in the mean and, if for any $\varepsilon > 0$, $t_0 \in R^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_{0,} \varepsilon)$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta_0$$

implies

$$E[||x(t,\omega)||] < \varepsilon, \ t \geq t_0+T.$$

Based on Definition 2, one can formulate other difinitions o stability and boundedness (see, [4,5,14]) analogously.

Defination 3. A function b (r) is said to belong to the clas K if $b \in C[R^+ + R^+]$, $b(0) = 0$, $b(r)$ is strictly increasing in r.

Definition 4. A function b(r) is said to belong to the class VK, if $b \in C\ [R^+, R^+]$, $b(0) = 0$, $b(r)$ is a convex and strictly increasing in r.

Definition 5. A function a(t,r) is said to belong to the class CK if $a \in C(R^+ X R^+, R^+]$, $a(t,0) = 0$, and $a(t,r)$ is concave and increasing in r for each fixed $t \in R^+$.

In order to avoid monotonicity, hereafter, it will be understood,unless otherwise specified, that the functions involved in (1)-(3) satisfy the conditions (iv) and (v) given in Section 1, and furthermore all equalities, inequalities, and relations that involve the random processes will hold with probability 1.

3. COMPARISON THEOREM

In this section, we develop a very general comparison theorem for stochastic integrodifferential system (1) which plays an important role in our subsequent discussion. In order to establish our main result in this section, we require the following hypothesis:

(H) the solution $y(t,t_0,x_0,\omega)$ of (2) exist for all $t \geq t_0$, unique, continuous with respect to the initial data and $||y(t,t_0,X_0,\omega)||$ is locally Lipschitzian in x_0.

THEOREM 1. *Assume that*

(i) *for each* $t \in R^+$, $x \in X$,

(4) $$\lim_{h\to 0^+} R(h,A(t,\omega))x = x,$$

(ii) *the hypothesis (H) holds,*

(iii) there exists a measurable map $V(t,x,\omega)$ defined on $R^+ x\ X \times \Omega$ into R^+ such that $V(t,x,\omega)$ is locally Lipschitzian in $(t,x) \in R^+ x\ X$ ω.p.1,

(iv) *for* $t \in R^+$, $x \in X$ and $\omega \in \Omega$,

(5) $$D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$= \lim_{h\to 0^+}\sup \frac{1}{h}\,[V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$

$$+h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,w)$$

$$+ g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$- V(s,y(t,s,x(s,\omega),\omega),\omega)]$$

$$\leq H_1(s,V(s,y(t,s,x(s,\omega),\omega),\omega),\int_{t_0}^{s} K_0[s,\tau,V(\tau,Y(t,\tau,x(\tau,\omega),\omega)\omega]$$

$$d\tau,\omega) + H_2(s,V(s,y)(t,s,x(s,\omega),\omega),\omega),\omega),$$

where $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ *and* $H_1(t,r,\bar{r},\omega)$ are as defined in section 1,

(v) the maximal solution $r(t,t_0,r_0\omega)$ of (3) exists for $t \geq t_0$. Then $x(t,\omega) = x(t,t_0,x_0,\omega)$ is any solution of (1) we have

(6) $$V(t,x(t,\omega),\omega) \leq r(t,t_0,r_0,\omega),\ t \geq t_0,$$

provided

(7) $V(t_0,y(t,t_0,x_0,\omega),\omega) \le r_0(\omega)$.

Proof. Define

$$m(s,\omega) = V(s,y(t,s,x(s,\omega),\omega),\omega) \text{ for } t_0 \le s \le t,$$

so that

$$m(t_0,\omega) = V(t_0,y(t,t_0,x_{0,}\omega),\omega).$$

Since $x(t,\omega)$ is a sample solution of (1) and $V(t,x,\omega)$ satisfies hypothesis (iii), we conclude that $m(s,\omega)$ is sample absolutely continuous W.p.1 for $t \ge t_0$ (see [4] and a particular reference given therein). For small $h > 0$, we have

(8) $m(s+h,\omega)-m(s,\omega)$

$$\begin{aligned}
&= V(s+h,y(t,s+h,x(s+h,\omega),\omega),\omega) - V(s,y(t,s,x(s,\omega),\omega),\omega)\\
&= V(s+h,y(t,s+h,x(s+h,\omega),\omega),\omega -V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)\\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)\\
&\quad + V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)\\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s}K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)\\
&\qquad -V(s,y(t,s,x(s,\omega),\omega),\omega)\\
&\le L||y(t,s+h,x(s+h,\omega),\omega)-y(t,s+h,R(h,A(s,\omega))x(s,\omega)\\
&\quad +h\{f(s,x(s,\omega),\int_{t_0}^{s}K[s,\tau,x(\tau,\omega),\omega)]d\tau,\omega)+g(s,x(s,\omega),\omega)\},\omega)||\\
&\quad + V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)+h\{f(s,x(s,\omega),\\
&\quad \int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)
\end{aligned}$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$\leq LM||x(s+h,\omega) - R(h,A(s,\omega))x(s,\omega) - h\{f(s,x(s,\omega)),$$

$$\int_{t_0}^{s}K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\}||$$

$$+V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega) + h\{f(s,x(s,\omega)),$$

$$\int_{t_0}^{s}K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega),$$

where L and M are positive constants. Since $x(t,\omega) \in D[A(t,\omega)]$, we have

$$R(h,A(t,\omega))\ [I-hA(t,\omega)]x(t,\omega) = x(t,\omega).$$

Hence

$$(9)\quad R(h,A(t,\omega))x(t,\omega) = x(t,\omega) + hA(t,\omega)x(t,\omega)$$
$$+h[R(h,A(t,\omega))A(t,\omega)x(t,\omega)-A(t,\omega)x(t,\omega)].$$

Using (9) in (8) we have

$$(10)\quad m(s+h),\omega\ )-m(s,\omega\ ) \leq LM||\ x(s+h,\omega)-x(s,\omega)-hA(s,\omega)x(s,\omega)$$
$$-h\ [R(h,A(s,\omega))A(s,\omega)x(s,\omega)-A(s,\omega)x(s,\omega)]\ \ -h\ \{f(s,x(s,\omega),$$
$$\int_{t_0}^{s} K\ [\ s,\tau,x\ (\tau,\omega),\omega,\ \omega)\ +g(s,x(s,\omega),\omega)\}||+V(s+h,y(t,s+h,$$
$$R(h,A(s,\omega))\ x(s,\omega)+h\ \{f(s,x(s,\omega),\ \ \int_{t_0}^{s}\ K\ [s,\tau,x(\tau,\ \omega),\omega)]d\tau,\omega)$$
$$+g(\ s,x(s,\omega)\omega\},\ \omega),\ \omega),-\ V(s,y(t,s,x(s,\omega),\omega),\omega).$$

Now, using (10),(1) and the assumption given in the statement of the theorem, and sample absolute continuity of $m(s,\omega)$ yeilds the inequality

$$(11)\ m'(s,\omega) \leq H_1(s, m(s,\omega), \int_{t_0}^{s} K_0[s,\tau,m(\tau,\omega)\ \omega]d\tau, \omega\ )$$

$$+ H_2(s,m(s,\omega),\omega)$$

almost everywhere on $(s,\omega) \in R^+ X \Omega$. From (7) and the definition of $m(s,\omega)$ we have $m(t_0,\omega) \leq r_0\ (\omega)$. Now by following the standard argument analogous to deterministic Theorem 1 in Ref [13, p.192] with suitable modification yields

$$(12)\ m(s,\omega) \leq r(s,t_0,r_0,\omega),\ t_0 \leq s \leq t$$

Provided $m(t_0,\omega) \leq r_0(\omega)$. Since

$$m(t,\omega) = V(t,y(t,t,x(t,\omega),\omega),\omega) = V(t,x(t,\omega),\omega),$$

the desired result (6) follows from (12) by setting s = t . The proof is complete.

It is interesting to make the following observations ·

(a) Taking $r_0(\omega) = V(t,t_0,x_0,\omega),\omega)$, the inequality (6) becames

$$(13)\ V(t,x(t,\omega),\omega) \geq r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega), \quad t \geq t_0,$$

which shows the connection between the solutions of systems (1) and (2) in terms of the maxial solutions of (3).

(b) The trivial function

$$A(t,\omega)y(t,\omega) + f(t,y(t,\omega), \int_{t_0}^{t} K\ [t,s;y(s,\omega),\omega\ ]ds,\ \omega) = 0$$

is admissible in Theorem 1 to yield the estimate (6) provded $V(t_0,x_0,\omega) \leq \gamma_0(\omega)$. In this case $y(t,t_0,x_0,\ \omega) = x_0$ and thus the hypothesis (H) is trivially verified. Since $y(t, t, x, \omega) = x$, the difinition (5) of $D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$ reduces to

$$(14)\ D^+V(t,x(t,\omega),\omega),\omega) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,R(h,A(t,\omega)x(t,\omega)$$

$$+h\{f(t,x(t,\omega), \int_{t_0}^{t} K[t,s,x(x,\omega),\omega]ds, \omega)$$

$$+ g(t,x(t,\omega),\omega)\},\omega)-V(t,x(t,\omega),\omega)\ ],$$

which is the usual definition of the generalized derivative of the Lyapunov function relative to the system (1).

We note that Theorem 1 is a direct eextension to stochastic integrodiffeerential systems of the corresponding comparision theorem for deterministic equations recently established by the present author in [13, Theorem 3] which in turn is a further extension of the comparison theorem established by Ladde, Lakshmikantham and Leela in [6]

4.STABILITY RESULTS

In this section employing the comparison theorem developed in the preceding section, we shall present various results giving sufficient conditions for stability in probability and in the mean of the trivial solution of the stockastic integro-differential system (1). Before giving the main results in this section, relative to the stochastic integrodifferential system (2) and the auxiliary stochastic integrodifferential equation (3) we need the corresponding definitions (CSP_1^*) (CSP_2^*),$(CSM_1^*)$and (CSM_2^*) in our discussion.

For example the connective stability in probability (CSP*) runs as follows (see, [6], for a similar definition in deterministic case).

Definition 6: The trivial solutions of (2) and (3) are said to be (CSP$_1^*$) connectively stable in probability, if for each $\varepsilon > 0$, $\eta >. 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0;\varepsilon\ \eta) > 0$ such that

$$P\ \{\omega:\ ||x_0\ (\omega)\ ||\ >\ \delta\}\ <\ \eta$$

implies

$$P\ \{\omega:\ r(t,\ t_0,\ v(t_0,y(t,t_0,x_0,\omega),\omega),\omega\ \geq\ \varepsilon\}\ <\ \eta, t\ \geq\ t_0.$$

Other definitions may similarly be formulated.

THEOREM 2. *Assume that the hypotehses of Theoreem 1 hold and*

(15) $b(||x||)\ \leq\ V(t,x(t,\omega),\omega)\ \leq\ a(t,\ ||x||)$,

where $b,\ a(t,.) \in \mathcal{K}$. *Then*

(i) (CSP$_1^*$) *implies* (SP$_2$),

(ii) (CSP$_2^*$) *implies* (SP$_1$)

Proof. First we prove the statement (i). Let $\eta > 0$, $\varepsilon > 0$, and $t_0\ \varepsilon\ R^+$ be given. Assume that (CSP$_1^*$) holds. Then, given $b(\varepsilon) > 0$, $\eta > 0$, and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon,\eta)$ such that

(16) $P\ \{\omega: r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\}\ <\eta, t\ \geq\ t_0$

provided

(17) $P\{\omega\ :||x_0(\omega)||\ >\ \delta_1\}\ <\ \eta$.

Let us choose $r_0(\omega)$ so that $V(t_0,y(t,t_0,x_0,,\omega),\omega) = r_0(\omega)$ and

$$r_0(\omega) = a(t_0,||\ x_0(\omega)||\ )$$

Since $a(t_0,\cdot) \in \mathcal{K}$ wee can find a $\delta = \delta(t_0,\epsilon\eta) > 0$ such that $P\{\omega : a(t_0,\|x_0(\omega)\|) > \delta_1\} = P\{\omega : \|x_0(\omega)\| > \delta\}$

Now, we claim that (SP_1) holds. Suppose that this claim is false . Then, there would exist a solution process $x(t,\omega)$ of (1) with $P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$ and a $t_1 > t_0$ such that

(18) $P\{\omega : \|x(t_1,\omega)\| \geq \varepsilon\} = \eta.$

By theorem 1, the inquality

(19) $V(t,x(t,\omega),\omega) \leq r(t,t_0,V(t_0,x_0,\omega),\omega),\omega)$

is valid for all $t \geq t_0, \omega \in \Omega$. From (15) and (19) , we have

(20) $b(\|(t,\omega)\|) \leq V(t,x(t,\omega),\omega)$

$$\leq r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)$$

The relations (16), (18) and (20) lead us to the contradiction

$$= P\{\omega : \|x(t_1,\omega)\| \geq \varepsilon\}$$

$$= P\{\omega : b(\|x(t_1,\omega)\| \geq b(\varepsilon)\}$$

$$\leq P\{\omega : r(t,t_0,V(t_0, y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta,$$

thus proving the statement (i).

Now, we shall give the proof of the statement (ii). Assume that (CSP_2^*) holds. Then given $b(\varepsilon) > 0, \eta > 0$, and $t_0 \in R^+$, there exist positive numbers $\delta^0(t_0) = \delta^0$ and $T = T(t_0,\varepsilon,\eta)$ such that(21) $P\{\omega : r(t,t_0 V(t_0,y(t_0 y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta, t \geq t_0+T$ whenever

$P\{\omega : \|x_0(\omega)\| > \delta^0\} < \eta.$

To prove statement (ii) , it is enough to prove that for any $\varepsilon > 0$, $\eta > 0$ and $t_0 \in R^+$ there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon,\eta)$ such that $P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$ implies

(22) $P\{\omega : \| x(t,\omega) \| \geq \varepsilon \} < \eta, \quad t \geq t_0 + T.$

We claim that (22) holds, otherwise, there exists a sequence $\{t_n\}, t_n \geq t_0 + T, t_n \to \infty$ as $n \to \infty$ such that for some solution process of (1) satisfying $P\{ \omega: \| x_0(\omega) \| \delta_0\} < \eta$, it will satisfy the relation

(23) $P\{\omega: \| x(t_n, \omega) \| \geq \varepsilon \} = \eta , \quad t_n \geq t_0 + T.$

This together with (19) and (21) will establish the vaalidity of (22) . This completes the proof of the theorem.

Our next theorem establishes the properties of (1) in the mean

Theorem 3 : Assume that the hypotheses of Theorem 1 hold and (15) *hold with* $b \in V\kappa$, $a \in C\kappa$ then

(i) (CSM_1^*) *implies* (SM_1)

(ii) (CSM_2) *implies* (SM_2).

Proof

Let us first prove the statement (i). Let $\varepsilon > 0, t_0 \in R^+$ be given. Assume that (CSM_1^*) holds. Then for given $b(\varepsilon) > 0$ and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon)$ such that $E[\| x_0(\omega) \|] \leq \delta_1$ implies

(24) $E[r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)] < b(\varepsilon) , t \geq t_0.$

we choose $r_0(\omega)$ such that $V(t_0,y(t,t_0,x_0,\omega),\omega) = r_0(\omega)$ and

$$E[\| x_0(\omega)\|] = a(t_0, E[\| x_0(\omega) \|]).$$

Since $a \in C\kappa$, we can find a $\delta = \delta(t_0,\in) > 0$ such that

$$E[\|x_0(\omega)\|] \leq \delta \text{ implies } a(t_0,E[\|x_0(\omega)\|]) < \delta_1$$

Now we claim that if $E\,[||x_0(\omega)||] \leq \delta$, then $E[||x(t,\omega)||] < \varepsilon$, $t \geq t_0$. Suppose that this is false. Then , there would exist a solution process $x(t,\omega)$ of (1) with $E\,[\,||x_0)\omega)||] \leq \delta$ and a $t_1 > t_0$ such that

(25) $E\,[||\,x(t_1,(\omega)\,||] = \in$ and $E\,[||\,x(t,\omega)||\,] \leq \in, t \in [t_0, t_1]$.

By Theorem 1 and (15) we have the inequality (20), and by the convexity of b we have

(26) $b(E[\,||x(t,\omega)\,||]) \leq E\,[\,V(t,x(t,\omega),\omega)]$

$\leq E\,[\,r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)]$

The relations (24), (25) and (26) lead to the contradiction

$b(\varepsilon) \leq E[V(t_{1,} x(t_1,\omega),\omega)\,]$

$\leq E[r(t,t_0,V(t_0,y(t_1,t_0,x_0,\omega),\omega),\omega)] < b\,(\in)$,

Providing (i). The proof of (ii) can be formulated similarily .

In concluding this section we note that, one can very easily formulate other definitions and theoremms on stability and boundedness as in [4,5,14] analogously with suitable modifications for the integrodifferental system (1). We omit the details.

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ICHTHYOFAUNA OF THE RIVER GANGA AT THE FOOTHILLS OF GARHWAL HIMALAYA

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ABSTRACT

The paper deals with the Ichthyofauna of the river Ganga at foothills of Garhwal Himalaya. A systematic list of 44 species is given with their present valid names and earlier names given by Day and Hamilton.

Key words and Phrases : Ichthyofauna, Ganga river, Systematic list.

INTRODUCTION

In 1822 Hamilton [5] listed the fish fauna of the river Ganga and its tributaries, no other work has been published before this, Day [4], Hora [7], Hora and Mukurjee [8], Jayaram [9], Mahajan [10], Pant [12] and Venkateshwarlu and Menon [15] have also published reports on the Ichthyofauna of the Ganga river system.

The fish fauna of this river in the uplands have been studied by Heckel [6], Malik [11], Sehgal et al.[13] Badola [1], Badola and Pant [2], Badola and Singh [3] and Singh et al.[14]. But the reports on the fish fauna of the Ganga river at the foothills of Garhwal Himalaya are scanty and limited. The paper deals with the Ichthyofauna of the river Ganga and its tributaries (Song and Suswa) from Lakshman Jhula to Kangri Village, Hardwar (Table).

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TABLE : SYSTEMATIC LIST OF FISHES

Present Scientific Name	Name according to Day	Name according to Hamilton	Local Name
A	B	C	D
	Phylum :	VERTEBRATA	
	Subphylum :	CRANIATA	
	Super class :	GNATHOSTOMATA	
	Series :	PISCES	
	Class :	TELEOSTOMI	
	Sub class :	ACTINOPTERYGII	
	Order :	CYPRINIFORMES	
	Sub order :	CYPRINOIDEI	
	Family :	**CYPRINIDAE**	
	Sub family :	**RASBORINAE**	
	Genus :	*Barilius* Hamilton	
Barilius bendelisis (Ham.)	*Barilius bendelisis*	*Cyprinus (Barilius) bendelisis*	Bhola
Barilius bola (Ham.)	*Barilius bola*	*Cyprinus (Barilius) Bola*	Nayar Bhola
Barilius vagra (Ham.)	*Barilius vagra* *Barilius modestus*	*Cyprinus (Barilius) vagra*	persee
	Genus : *Danio*	Hamilton	
Danio (Danio) devario (Ham.)	*Danio devario*	*Cyprinus (Babdio) devario*	Patukari
Danio (Brachydanio) rerio	*Danio rerio*	*Cyprinus (Danio) rerio* *Cyprinus (Danio) chapalio*	
	Genus : *Esomus* :	Swainson	
Esomus danricus (Ham.)	*Nuria danrica*	*Cyprinus (Danio) danrica* *Cyprinus (Danio) sunita* *Cyprinus (danio) jogia*	Denc

Contd..

A	B	C	D
	Genus : *Rasbora* Bleeker		
Rasbora daniconius (Ham.)	*Rasbora daniconius*	*Cyprinus* (*Danio*) *daniconius* *Cyprinus* (*Danio*) *anjana*	Dendua
	Sub family : Cyprininae		
	Ganus : *Chagunio* Smith		
Chagunio chagunio (Ham.)	*Barbus chagunio*	*Cyprinus* (*Cyprinus*) *chagunio*	Gelhari
	Genus : *Crossocheilus* Van hasselt		
Crossocheilus latius latius (Ham.)	*Cirrhina latia*	*Cyprinus* (*Garra*) *latius* *Cyprinus* (*Garra*) *gohama* *Cyprinus* (*Garra*) *sada*	Pet phorni
	Genus : *Garra* Hamilton		
Garra gotyla gotyla (Gray)	*Discognathus lamta* impart	*Cyprinus lamta*	Siltoka
	Genus : *Labeo* Cuvier		
Labeo boga (Ham.)	*Labeo boga*	*Cyprinus* (*Bangana*) *boga*	Bhagan
Labeo dero (Ham.)	*Labeo microphthalmus* *Labeo diplostomus* *Labeo sindensis*	*Cyprinus* (*Bangana*) *dero*	Arangi
Labeo gonius (Ham.)	*Labeo gonius*	*Cyprinus* (*Cyprinus*) *gonius*	
		Cyprinus (*Cyprinus*) *curchiu*	kursi
		Cyprinus (*Cyprinus*) *cursa*	Bata

A	B	C	D
	Genus : *Puntius* Hamilton		
Puntius sarana sarana (Ham.)	*Barbus sarana* *Barbus chrysopoma* *Barbus pinnauratus*	*Cyprinus* (*Cyprinus*) *Saran*	Darahee
Puntius sophore (Ham.)	*Barbus sophore* *Barbus stigma* *Barbus puntio* *Barbus chrysopterus*	*Cyprinus* (*puntius*) *sophore* *Cyprinus* (*puntius*) *puntio*	Sidhari
Puntius ticto (Ham.)	*Barbus ticto*	*Cyprinus* (*puntius*) *ticto*	Sidhari
	Barbus punctatus		
	Genus : *Tor* Gray		
Tor putitora	*Barbus tor* (in part	*Cyprinus* (*Cyprinus*) *putitora*	Mahaseer
Tor tor (Ham.)	*Barbus tor* (in part) *Barbus hexasticus*	*Cyprinus* (*Cyprinus*) *tor*	Mahaseer
	Sub family : **Schizothoracinae** Heckel		
	Genus : *Schizothorax* Heckel		
Schizothorax plagiostomus	*Oreinus plagiostomus*		Dhibrua
Schizothorax sinuatus	*Oreinus sinuatus*		Maseen

Contd..

A	B	C	D
	Genus : *Schizothoraichthys* Misra		
Schizothoraichthys progastus (mc Clelland)	*Schizothorax progastus*		Chonga
	Family : **Cobitidae**		
	Genus : *Botia* Gray		
Botia dario (Ham.)	*Botia dario*	*Cabitus geto*	Baghaua
	Genus : *Nemacheilus* Van Hasselt		
Nemacheilus beavani (*Gunther*)	*Nemacheilus beavani*		Natwa
Nemacheilus botia (Ham.)	*Nemacheilus botia*	*Cobitus botia* *Cobitus bilturio*	Natwa
Nemacheilus montanus (Mc Clelland)	*Namacheilus denisoni*		Natwa
Nemacheilus rupicola (Mc Clelland)	*Nemacheilus rupicola*		Natwa
Nemacheilus savona	*Nemacheilus savona*	*Cobitus turio*	Natwa
	Division : SILURI SUB order ; SILUROIDEI Family : Bagridae		
	Genus : *Mystus* Gronow (emend. Scopoli)		
Mystus cavasius (Ham.)	*Macrones cavasius*	*Pimelodus cavasius*	Sutahawa Tengara
Mystus tengara (Ham.)	*Macrones tengara*	*Pimelodus tengara*	Tengana
Mystus seenghala (Sykes)	*Macrones seenghala*		Seenghara

A	B	C	D
	Genus : *Rita* Bleeker		
Rita rita (Ham.)	*Rita buchanani*	*Pimelodus rita*	Hunna Rita
	Family: **Sisoridae** Genus : *Bagarius* Bleeker		
Bagarius bagarius	*Bagarius yarrelli*	*Pimelodus bagarius*	Gonch
	Genus : *Glyptothorax* Blyth		
Glytothorax conirostris (*Steind*)	*Glyptosternum conirostri* *Glyptosternum lonah* (in part)		Nau
Glyptothorax pectinopterusrus (Mc Clelland)	*Glyptosternum pectinopterum*		Nau
Glyptothorax madraspatanum	*Glyptosternum modestum*		Nau
	Family : **Schilbeidae** Genus : *Clupiscma* Swainson		
Clupisoma garua (Ham.)	*Pseudeutropius garua*	*Silurus* (*Callichrous*) *garua*	Baikari Karahi
	Order : BELONIFORMES Suborder: SCOMERBOSOCOIDEI Family : **Belonidae** Genus : *Xenentodon* Regan		
Xenentodon cancila (Ham.)	*Belone cancila*	*Esor cancila*	*Kauva*

Contd..

A	B	C	D
	Order : OPHIOCEPHALIFORMES (CHANNIFORMES) Family : **Ophiocephalidae (Channidae)** Genus : *Channa* Scopoli		
Channa marulius (Ham.)	*Ophiocephalus marulius*	*Ophiocephalus marulius*	Saur
Channa punctatus (Bloch)	*Ophiocephalus punctatus*	*Ophiocephalus lata*	Girai
Channa striatua (Bloch)	*Ophiocephalus striatus*	*Ophocephalus wrahl* *Ophiocephalus chena*	Saur
Channa gachua (Ham.)	*Ophiocephalusgachua*	*Ophiocephalus gachua*	Chanaga
	Order : PERCIFORMES Suborder: ANABANTOIDEI Family : **Anabantidae** Ganus : *Colisa* Cuvier et valenciennes		
Colisa fasciatus(Block et schn.)	*Trichogaster fasciatus*	*Trichopodus colisa* *Trichopodius bejeus* *Trichopodius cotra*	Khosti
	Order : MASTACEMBELIFORMES Family : **Mastacembelidae** Genus : *Mastacembelus* Schopoli		
Mastacembelus armatus (Lacepede)	*Mastacembelus armatus*	*Macrognathus armatus*	Baam
Mastacembelus pancalus (Ham.)	*Mastacembelus pancalus*	*Macrognathus pancalus*	Malga Pataya

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COINCIDENCE THEOREMS IN LINEAR TOPOLOGICAL SPACES

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ABSTRACT

The notion of diminishing joint orbital diametral sum for a pair of commuting mapping has been generalized to a Hausdorff locally convex space whose topology is generated by a family of seminorms. Subsequently, this notion is utilized to prove some coincidence theorems which generalize several known results.

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1. INTRODUCTION

Several fixed point theorems for nonexpansive mappings in locally convex linear topological spaces are known. In this setting the notions of diminishing orbital diameters and normal structure were given by Tan [7]. These notions were the generalizations of the corresponding notions given earlier by Belluce and Kirk [1] and Brodskii and Milman [2]. Recently the notions of diminishing joint orbital diameteral sum and joint normal structure in a metric space were introduced by Mishra [5] and several coincidence theorems for a pair of commuting mappings on metric and Banach spaces were proved. In this paper it is shown that

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most of these results hold for the same class of mappings on a bounded subset of a Hausdorff locally convex linear topological space.

Let (E, τ) be a locally convex linear topological space. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is said to be an associated family of seminorms for τ if the family $B = \{tU : t > 0\}$ where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x \in E : p_{\alpha_i}(x) < 1\}$ forms a base of neighbourhoods of zero for τ. The set U is also given by $U = \{x \in E : p(x) < 1\}$ where p is the seminorm max $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is called an augmented associated family of seminorms for τ if it is an associated family for τ and has further property that given $\beta, \gamma \in \Lambda$ the seminorm max $\{p_\beta, p_\gamma\} \in \{p_\alpha : \alpha \in \Lambda\}$. We shall denote an associated family and an augmented associated family for τ by D and D* respectively. It is well-known that given a locally convex linear topological space (E, τ), there always exists a family of seminorms $\{p_\alpha : \alpha \in \Lambda\}$ such that $\{p_\alpha : \alpha\ \Lambda\} = D^*$. For details we refer to Köthe [4]. Unless otherwise stated, throughout this paper, E will denote a Hausdorff locally convex linear topological space (E, τ) defined by $\{p_\alpha : \alpha \in \Lambda\} = D^*$.

DEFINITION 1.1 [9]. Let M be a nonempty subset of E. A mapping $f : M \to M$ is called D-nonexpansive (resp. D*-nonexpansive) if for all $x, y \in M$ and for each $p_\alpha \in D$ (resp. for each $p_\alpha \in D^*$), $p_\alpha(f(x) - f(y)) \leq p_\alpha(x-y)$.

Since f is D-nonexpansive $\Leftrightarrow$ f is D*-nonexpansive (cf. Tarafdar [8]), we shall simply say f is nonexpansive. For equivalent difinitions of nonexpansive mappings, we refer to Srivastava and Srivastava [6], Tan [7] and Taylor [10].

Let S and T be commuting mappings of E into E, and let $f = ST = TS$. For each $x \in E$, let $0(f^n(x)) = \{f^n(x), f^{n+1}(x), \ldots\}$ where $n = 0, 1, 2, \ldots$ and $f^0(x) = x$. For $x, y \in E$ and $\alpha \in \Lambda$, define

$$p_\alpha(x,y) = p_\alpha(Tx-Sx) + p_\alpha(Ty-Sy) + p_\alpha(Tx-Sy) + p_\alpha(Ty-Sx) + p_\alpha(Tx-Ty).$$

It is clear that $P_\alpha(x,y) = p_\alpha(y,x)$. For each nonempty subset A of E, define $J_\alpha(A) = \sup\{P_\alpha(x,y) : x, y \in A\}$. Further, suppose that $R_\alpha(f^n(x)) = J_\alpha(0(f^n(x)))$ and $r_\alpha(x) = \lim\limits_n R(f^n(x))$. Then we have the following:

DEFINITION 1.2. If for each $x \in E$, there is an $\alpha \in \Lambda$ such that $J_\alpha(0(x)) < \infty$ and $r_\alpha(x) < J_\alpha(0(x))$ whenever $J_\alpha(0(x)) > 0$, then mappings $S, T : E \to E$ will be said to have diminishing joint orbital diametral sum (dimjodis) on E.

DEFINITION 1.3.(i) Let A be a bounded subset of E. A point $a \in A$ will be called a joint nondiametral point of A (with respect to $S, T : E \to E$) if there is an $\alpha \in \Lambda$ such that $\sup\{p_\alpha(a,x) : x \in A\} < J_\alpha(A)$.

(ii) A bounded subset K of E will be said to have joint normal structure (with respect to $S, T : E \to E$) if for each convex subset H of K which contains more than one point, there is a joint nondiametrial point of H.

For equivalent definitions in metric spaces, we refer to [5].

Let K be a nonempty subset of E. A mapping $g : K \to K$ is called affine if $g(kx+(1-k)y) = k\,gx + (1-k)y$ for all $x, y \in K$ and for all $k \in [0,1]$.

PROPOSITION 1.4. Let S, T : $E \to E$, and let A, $B \subseteq E$ such that $A \subseteq B$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) \le J_\alpha(B)$,

(ii) $J_\alpha(A) = J_\alpha(\bar{A})$, if S and T are continuous, where $\bar{A}$ is the colsure of A.

PROOF. The proof of (i) is obvious. We shall prove (ii). Let $\alpha \in \Lambda$ be arbitrary. Since $A \subseteq \bar{A}$ it follows that $J_\alpha(A) \le J_\alpha(\bar{A})$. To prove that $J_\alpha(\bar{A}) \le J_\alpha(A)$, let $x, y \in \bar{A}$, and let $\epsilon > 0$. Since S and T are continuous, we may choose u,v in A such that $p_\alpha(Tx-Tu) < \epsilon$, $p_\alpha(Ty-Tv) < \epsilon$, $p_\alpha(Sx-Su) < \epsilon$, and $p_\alpha(Sy-Sv) < \epsilon$. Hence using the definition of $p_\alpha(x,y)$ and triangle inequality, it can be easily verified that

$$P_\alpha(x,y) \le P_\alpha(u,v) + 3\,(p_\alpha(Tx-Tu) + p_\alpha(Ty-Tv)) + 2(p_\alpha(Sx-Su) + p_\alpha(Sy-Sv))$$
$$\le P_\alpha(u,v) + 10\,\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $P_\alpha(x,y) \le P_\alpha(u,v)$. Hence $J_\alpha(\bar{A}) \le J_\alpha(A)$, proving (ii).

PROPOSITION 1.5: Let A be a subset of E, and let S, T: $E \to E$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) = J_\alpha(A)_c$, if S and T are affine,

(ii) $J_\alpha(A) = J_\alpha\overline{(A)}_c$, if S and T are affine and continuous; where $(A)_c$ and $\overline{(A)}_c$ are the convex hull and closed convex hull of A respectively.

PROOF. We shall prove (i) as (ii) follows from (i) and 1.4 (ii). Since $A \subseteq (A)_c$ we have from 1.4 (i) that $J_\alpha(A) \le J_\alpha(A)_c$. To prove that $J_\alpha(A)_c \le J_\alpha(A)$, let $x, y \in (A)_c$. Then we may assume that $\Lambda = \Sigma\, a_i x_i$ and $y = \Sigma\, b_j x_j$

and $y = b_j x_j$ where a_i, $b_j > 0$, $\sum a_i = 1 = \sum b_j$ and $x_i, x_j \in A$, $i=1,2, \dots, n$ and $j = 1,2, \dots, n$. Then following [5] it can be shown that

$$p_\alpha (Tx-Sx) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_i - Sx_i)\ ,$$

$$p_\alpha (Ty-Sy) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_j - Sx_j)\ ,$$

$$p_\alpha (Tx-Sy) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_i - Sx_j)\ ,$$

$$p_\alpha (Ty-Sx) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_j - Sx_i),$$

$$p_\alpha (Tx-Ty) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_i - Tx_j)$$

for each $\alpha \in \Lambda$.

Therefore $P_\alpha(x,y) \leq (\Sigma\, b_j)\ (\Sigma\, a_i\, P_\alpha(x_i,x_j)) \leq (\Sigma b_j)$ $(\Sigma\, a_i)\, J_\alpha (A)$. Hence $J_\alpha (A)_c \leq J_\alpha (A)$. This proves that $J_\alpha (A) = J_\alpha (A)_c$.

2. MAPPINGS WITH DIMJODIS

In this section we shall prove two coincidence theorems for mappings with diminishing joint orbital diametral sum (dimjodis) and with normal structure. The results obtained herein generalize certain theorems of Mishra [5, Theorems 3.2 and 3.3] and Tan [7].

DEFINITION 2.1. Let $a \in E$. Define $B_\alpha (a,r) = \{x \in E : P_\alpha(a,x) \leq r\}$, $r > 0$ and $\alpha \in \Lambda$.

THEOREM 2.2. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that

(i) $\overline{O(x)_c} \cap M \neq \phi$ for each $x \in K$, and

(ii) S and T have *dimjodis* on K,

then there exists a $z \in M$ such that $Sz = Tz$ i.e. S and T have a coincidence point z in M.

PROOF. Let $f = ST = TS$. A nonempty subset K^* of K minimal with respect to being closed convex f-invariant and having points in common with M is obtained by using weak compactness of M and the Zorn's lemma. Let $M^* = K^* \cap M$.

Let $x \in K^*$ and $\alpha \in \Lambda$. Then we wish to show that $J_\alpha(O(x)) = 0$. Suppose $J_\alpha(O(x)) > 0$. By (ii), there exists an integer N such that $k_\alpha = J_\alpha(O(f^n(x))) < J_\alpha(O(x))$ Suppose $L = \{z \in K^* : p_\alpha(z, f^N(x)) \leq k_\alpha$ for almost all $n\}$. Then $L \subseteq K^*$. We shall show that L is nonempty convex invariant under f and has points in common with M. Taking $f^m(x) \in O(f^N(x))$, i.e. $m \geq N$ we have $p_\alpha(f^m(x), f^n(x)) \leq k_\alpha$ for all $n, m \geq N$. Hence $f^m(x) \in L$ and thus $L \neq \phi$. Suppose $y \in L$. Then for some intger N_1 we have $p_\alpha(y, f^n(x)) \leq k_\alpha$ for all $n \geq N_1$. Consider

$$\begin{aligned} P_\alpha(f(y), f^{n+1}(x)) = {} & p_\alpha(Tf(y) - Sf(y)) + p_\alpha(Tf^{n+1}(x) - Sf^{n+1}(x)) \\ & + p_\alpha(Tf(y) - Sf^{n+1}(x)) + p_\alpha(Tf^{n+1}(x) - Sf(y)) \\ & + p_\alpha(Tf(y) - Tf^{n+1}(x)) , \end{aligned}$$

Since f is nonexpansive (as S and T are so) and S and T are commuting, it follows that

$$\begin{aligned} P_\alpha(f(y), f^{n+1}(x)) \leq {} & p_\alpha(Ty - Sy) + p_\alpha(Tf^n(x) - Sf^n(x)) \\ & + p_\alpha(Ty - Sf^n(x) + p_\alpha(Tf^n(x) - Sy) \\ & + p_\alpha(Ty - Tf^n(x)) \\ = {} & P_\alpha(y, f^n(x) \leq k_\alpha \end{aligned}$$

for all $n + 1 \geq N_1 + 1$. Hence $f(y) \in L$ and thus f maps L inot L. It follows from the continuity of f that $f(\bar{L}) \subseteq \overline{f(L)} \subseteq \bar{L}$. Therefore f maps L into L. Also $0(f^N(x)) \subseteq \bar{L}$, if follows that $0(f^N(x))_c \subseteq \bar{L}$. Therefore by (ii) we have $\bar{L} \cap M \neq \phi$. Hence the minimality of K* implies that $K^* = \bar{L}$.

Now let $u \in K^*$. Then since $u \in \bar{L}$, for $\epsilon > 0$ there exists a point u* inL such that $p_\alpha(u-u^*) < \epsilon/5$. Moreover, we can find an N_2 such that $p_\alpha(u^*, f^n(x)) \leq k_\alpha$ for all $n \geq N_2$. Further,

$$\begin{aligned} p_\alpha(u, f^n(x)) &= p_\alpha(Tu-Su) + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha(Tu-Sf^n(x)) \\ &\quad + p_\alpha(Tf^n(x)-Su) + p_\alpha(Tu-Tf^n(x)) \\ &\leq p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Su^*) + p_\alpha(Su^*-Su) \\ &\quad + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha(Tu-Tu^*) \\ &\quad + p_\alpha(Tu^*-Sf^n(x)) + p_\alpha(Tf^n(x) - Su^*) \\ &\quad + p_\alpha(Su^*-Su) + p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Tf^n(x)) \end{aligned}$$

No using the nonexpansiveness of S and T it can be easily verified that $p_\alpha(u, f^n(x)) \leq p_\alpha(u^*, f^n)) + 5\, p_\alpha(u-u^*) \leq k_\alpha + \epsilon$ for all $n \geq N_2$.

Let $B_\alpha(u, k_\alpha + \epsilon) = \{y \in E : p_\alpha(y,u) \leq k_\alpha + \epsilon\}$. Then we observe that $0(f^n(x)) \subseteq B_\alpha(u, k_\alpha + \epsilon)$ for all $n \geq N_2$. Since S and T are continuous and affine, $B_\alpha(u, t_\alpha + \epsilon)$ is closed and convex and hence $\overline{0(f^n(x))}_c \subseteq B_\alpha(u, k_\alpha + \epsilon)$ for all $n \geq N_2$. Also by (i) $\overline{0(f^n(x))}_c \cap M^* \neq \phi$ for al $n \geq N_2$. Hence the weak compactness of M* ensures the existence of a point t in $\bigcap_{n=N_2}^{\infty} \overline{0(f^n(x))}_c \cap M^*$ and consequently $t \in B_\alpha(u, k_\alpha + \epsilon)$. Since $\epsilon > 0$ and $u\ (\in K^*)$ are arbitrary, $t \in B_\alpha(u, k_\alpha)$

for all $u \in K^*$. Hence the set $A = \{z \in K^* : K^* \subseteq B_\alpha(u,k_\alpha)$ is nonempty. Since S and T are affine and continuous, $A, \subseteq K^*$ is closed and onvex.

We shall show that $f(A) \subseteq A$. Suppose this is not the case. Then $f(z) \notin A$ for some $z \in A$. Clearly $H = B_\alpha(f(z), k_\alpha) \cap K^*$ is closed and convex. Since $z \notin A$, we have $z \in K^* \subseteq B_\alpha(Z,k_\alpha)$. Also $f(z) \in K^*$. Hence $f(z) \in B_\alpha(z,k_\alpha)$. Therefore $z \in H$ and H is nonempty. Since $f(z) \notin A$, there exists $a \in K^*$ such that $y \notin B_\alpha(f(z),k_\alpha)$. This implies that $P_\alpha(f(z),y) > k_\alpha$. Hence H is a propφer subset of K^*. Let $y^* \in K^*$. Then $f(y^*) \in K^*$. Also $y^* \in K^* \subseteq B_\alpha(z,k_\alpha)$, it follows that $P_\alpha(y^*,z) \le k_\alpha$. It can be easily verified that $P_\alpha(f(z),f(y^*)) \le P_\alpha(z,y^*) \le k_\alpha$. Therefore $f(y^*) \in H$, proving $f(H) \subseteq H$. Moreover, since $z \in H$, $\overline{0(z)}_c \subseteq H$ and hence from (i) $H \cap M \neq \phi$. This shows that H is a nonempty closed convex proper subset of K^* which is invariant under f and has points in common with M, a contradiction to the minimality of K^*. Hence $f(A) \subseteq A$.

Let $v,w \in A$. Then $P_\alpha(v,w) \le k_\alpha$. Consequntly, $J_\alpha(A) \le k_\alpha = J_\alpha(0(f^N(x))) \le J_\alpha(0(x)) < J_\alpha(K^*)$. Therefore A is a proper subset of K^*. Hence our assumption that $J_\alpha(0(x)) > 0$ is incorrect. Therefore $J_\alpha(0(x)) = 0$ and $Sx = Tx$.

THEOREM 2.3. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that for each $x \in K$,

(i) $\overline{(0(x))}_c \cap M \neq \phi$, and

(ii) $\overline{(0(x))}_c$ has joint normal structure,

then there exists a point $z \in M$ such that $f(x)=z$, where $f = ST = TS$.

PROOF. Define K^* as in the proof of Theorem 2.2. We shall show that for each $x \in K^*$, $0(x)$ is a singleton. Suppose that there is an $x \in K^*$ such that $0(x)$ contains more than one point. Then by (ii), there exists a point $y \in \overline{(0(x))_c}$ such that y is a joint nondiametral point (with respect to S and T). That is there exists an $x \in \Lambda$ such that $\sup \{P_\alpha(y,z) : z \in \overline{(0(z))_c}\} = s_\alpha < J_\alpha(\overline{(0(x))_c}) = J_\alpha(0(x))\}$. Let $A = \{z \in K^* : K^* \subseteq B_\alpha(z, s_\alpha)\}$. It can be shown as in the proof of Theorem 2.2 that A is closed convex, $A \cap M \neq \phi$ and $f(A) \subseteq A$. By the minimality of K^* it follows that $A = K^*$. Hence $J_\alpha(K^*) = J_\alpha(A) \leq s_\alpha < J_\alpha(0(x)) \leq J_\alpha(K^*)$ which is absured. Hence for each $x \in K^*$, $0(x)$ is a singleton. Hence $f(x) = x$ for all $x \in K^*$.

3. BOUNDED MAPPINGS

In this section we shall discuss further properties of mappings with diminishing joint orbital diametral sum (dimjodis) by using the notion of bounded mappings. Such mappings were first studied by Kirk [3]. The term bounded mappings is due to Tan [7].

DEFINITION 3.1. Let $K \subseteq E$ be nonempty. A mapping $f : K \to K$ is said to be bounded if and only if for each $\alpha \in \Lambda$ there is a real number $c_\alpha > o$ such that $p_\alpha(f^k(x), f^k(y)) \leq c_\alpha p_\alpha(x-y)$ for all $x, y \in K$ and all $k = 1,2,\ldots$.

In the above condition if we set $f = ST = TS$, where S and T are self mappings of K and $c_\alpha = 1$, then f is nonexpansive and this happens when both S and T are nonexpansive.

PROPOSITION 3.2. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting continuous mappings of K into K such that $f = ST = TS$ is bounded. Then for each $\alpha \in \Lambda$ the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous.

PROOF. Let $\{x_\lambda : \lambda \in I\}$ be a net in K such that $x_\lambda \to x$ for some $x \in K$. Let I* be a directed subset (directed by $\leq$) of I. To prove that the given function is continuous we shall whow that $R_\alpha(x_\lambda) \to R_\alpha(x)$ for each $\alpha \in \Lambda$. Consider

$$|R_\alpha(x_\lambda) - R_\alpha(x)| = |\sup\,(P_\alpha(xy) : x,y \in O(x_\lambda) - \sup\{P_\alpha(x,y) : x,y \in O(x) \,|$$

Further,

$$\sup\{P_\alpha(x,y) : x,y, \in O(x_\lambda)\}$$

$$= \sup_{i,j \geq 0} \{P_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) + p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) + p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda)) + p_\alpha(Tf^i(x_\lambda)-Sf^i(x_\lambda)) + p_\alpha(Tf^i(x_\lambda)-Tf^j(x_\lambda))\},$$

$$\sup\{P_\alpha(x,y) : x,y \in O(x)\}$$

$$= \sup_{i,j \geq 0} \{p_\alpha(Tf^i(x) - Sf^i(x)) + p_\alpha(Tf^j(x)-Sf^j(x)) + p_\alpha(Tf^i(x) - Sf^j(x)) + p_\alpha(Tf^j(x) - Sf^i(x)) + p_\alpha(Tf^i(x) - Tf^j(x))\}.$$

Therefore,

$$|R_\alpha(x_\lambda) - R_\alpha(x)| \leq \sup_{i,j \geq 0} \{A_1+A_2+A_3+A_4+A_5\}$$

where

$$A_1 = |p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))|,$$

$$A_2 = |p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^j(x)) - Sf^j(x))|,$$

$$A_3 = |p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^i(x)) - Sf^j(x))|,$$

$$A_4 = |p_\alpha(Tf^j(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^j(x) - Sf^i(x))|,$$

$$A_5 = |p_\alpha(Tf^i(x_\lambda) - Tf^j(x_\lambda)) - p_\alpha(Tf^i(x) - Tf^j(x))|.$$

Let $\varepsilon > 0$ and set $\xi = \min\{\varepsilon/_{10}, \varepsilon/_{10c_\alpha}\}$. Since $x_\lambda \to x$ and S and T are continuous, there exists $\lambda_o \in I^*$ such that for all $\lambda \geq \lambda_o$ we have

$$(3.2.1) \qquad \begin{aligned} p_\alpha(x_\lambda - x) &< \xi, \\ p_\alpha(Tx_\lambda - Tx) &< \xi, \\ p_\alpha Tx_\lambda - Sx) &< \xi. \end{aligned}$$

Further,

$$\begin{aligned} p_\alpha &(Tf^i(x_\lambda) - Sf^i(x_\lambda)) \\ &\leq p_\alpha(Tf^i(x_\lambda) - Tf^i(x)) + p_\alpha(Tf^i(x) - Sf^i(x)) \\ &\quad + p_\alpha(Sf^i(x) - Sf^i(x_\lambda)), \end{aligned}$$

and

$$\begin{aligned} p_\alpha &(Tf^i(x_\lambda) - Tf^i(x)) \\ &= p_\alpha(f^i(Tx_\lambda) - f^i(Tx)) \leq c_\alpha p_\alpha(Tx - Tx), \\ p_\alpha &(Sf^i(x_\lambda) - Sf^i(x)) \leq c_\alpha p_\alpha(Sx - Sx). \end{aligned}$$

Therefore,

$$(3.2.2) \quad \begin{aligned} &|p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))| \\ &\leq c_\alpha\{p_\alpha(Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx)\}, \end{aligned}$$

..... Similiarly,

$$(3.2.3) \quad \begin{aligned} &|p_\alpha(Tx_\lambda - Sx_\lambda) - p_\alpha(Tx_\lambda - Sx)| \\ &\leq p_\alpha((Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx). \end{aligned}$$

Therefore from (3.2.1) through (3.2.3) it follows that $A_1 < \varepsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Similarly, $A_2 < \varepsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Using similar arguments, each of A_3, A_4 and A_5 can be made less than $\varepsilon/5$ for all i, j, ≥ 0 and for all $\lambda > \lambda_o$. Consequently,

$$|R_\alpha(x_\lambda) - R_\alpha(x)| < \epsilon \text{ for all } \lambda \geq \lambda_0 .$$

Therefore $R_\alpha(x_\lambda) \to R_\alpha(x)$.

COROLLARY 3.3. Let $K \subseteq E$ be nonempty bounded, and S and T be commuting nonexpansive mappings of K into K. Then the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous for each $\alpha \in \Lambda$.

THEOREM 3.4. Let $K \subseteq E$ be nonempty bounded, let S and T be commuting continuous mappings of K into K such that (3.4.1) S and T have *dimjodis*.

(3.4.2) f (= ST=TS) is bounded ,

(3.4.3) there is an $x \in K$ such that a subsequency of the sequence $\{f^n(x)\}$ converges to a point $z \in K$.

Then $\lim_n f^n(x) = z$ and z is a coincidence point of S ant T, i.e. $Sz = Tz$.

PROOF. Let $\{f^{n(s)}(x)\}$ be a subequence of $\{f^n(x)\}$ with its limit z ($z \in K$). Then since $R_\alpha(x)$ is continuous, for each $\alpha \in \Lambda$, it follows that $\lim_n R_\alpha(f^{n(s)}(x) = R_\alpha(z)$. Also $\lim_n R_\alpha(f^n(x)) = r_\alpha(x)$ and $\{R_\alpha(f^{n(s)}(x))\}$ is a subsequence of $\{R_\alpha(f^n(x))\}$, if follows that

(3.4.4) $\lim R_\alpha(f^{n(s)}(x)) = r_\alpha(x) = R_\alpha(z)$.

Since $\lim_n f^{n(s)}(x) = z$ and S and T are continuous, for $\epsilon > 0$ there exists a positive integer m such that for all $s \geq m$,

$$p_\alpha(f^{n(s)}(x) - z) < \epsilon/10\, c_\alpha ,$$

$$p_\alpha(f^{n(s)}(Tx) - Tx) < \epsilon/10\, c_\alpha ,$$

$$p_\alpha(f^{n(s)}(S\ ) - Sx) < \epsilon/10\, c_\alpha ,$$

Since f is bounded, we have for all $t \geq 1$,

$$(3.4.5)\quad p_\alpha(f^{n(s)+t}(Tx) - f^{t}(Tz)) < \epsilon/10$$

$$p_\alpha(f^{n(s)+t}(Sx) - f^{t}(Sz) < \epsilon/10.$$

Let N be a positive integer. Then

$$(3.4.6)\ R_\alpha(f^{n(s)+N}(x))$$

$$= \sup_{i,j \geq N} \{p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x))$$

$$p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+j}(x) +$$

$$p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+j}(x)) +$$

$$p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+i}(x)) +$$

$$p_\alpha(Tf^{n(s)+i}(x) - Tf^{n(s)+j}(x))\}.$$

Also,

$$p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x))$$

$$= p_\alpha(f^{n(s)+i}(Tx) - f^{n(s)+i}(Sx)$$

$$\leq p_\alpha(f^{n(s)+i}(Tx) - f^{i}(Tz)) + p_\alpha(f^{i}(Tz) - f^{i}($$

$$p_\alpha(f^{i}(Sz) - f^{n(s)+i}(Sx))$$

$$< p_\alpha(f^{i}(Tz) - f^{i}(Sz) + \epsilon/5 = p_\alpha(Tf^{i}(z) - Sf^{i}(z) + \epsilon$$

follows from (3.4.5).

Using a similar computation for other terms of

we have for $s \geq t$,

$$
\begin{aligned}
R_\alpha(f^{n(s)+N}(x)) < \sup_{i,j \geq N} \{ & p_\alpha(Tf^i(z) - Sf^i(z)) + \\
& p_\alpha(Tf^i(z) - Sf^j(z) + \\
& p_\alpha(Tf^i(z) - Sf^j(z)) + \\
& p_\alpha(Tf^j(z) - Sf^i(z)) + \\
& p_\alpha(Tf^i(z) - Tf^j(z)) \} + \epsilon
\end{aligned}
$$

$$= R_\alpha(F^N(z)) + \epsilon.$$

Hence $r_\alpha\ (x) = \operatorname{Lim}_N R_\alpha(f^{n(s)+N}(x)) \leq \lim_N R_\alpha(f^N(z))$

$$= r_\alpha(z) + \epsilon \text{ for all positive integers } N .$$

Since $\epsilon > 0$ is arbitrary, it follows that $r\ (x) \leq r_\alpha(z)$.

Hence from (3.4.4), we have

(3.4.7) $R_\alpha(z) = r_\alpha(x) \leq r_\alpha(z)$.

Supposr $R_\alpha(z) > 0$. Then since S and T have dimjodis, we have $r_\alpha(z) < R_\alpha(z)$, a contradiction to (3.4.7). Hence $R_\alpha(z) = 0$, proving $Sz = Tz$. Further, since $r_\alpha(x) = 0 = \lim_n r_\alpha(f^n(x))$, it follows that sequence $\{f^n(x)\}$ is a Cauchy sequence. Therefore $\lim_n f^n(x) = z$.

COROLLARY 3.5. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting nonexpansive mappings of K into K having dimjodis ; and if for some $x \in K$ a subsequence of the sequency $\{f^n(x)$, where $f = ST = TS$, has a limit z in K, then $\lim_n f^n(x) = z$ and $Sz = Tz$.

Since $K \sqsubseteq E$ is sequentially compact implies K is bounded, we have the following:

COROLLARY 3.6. Let $K \sqsubseteq E$ be nonempty sequentially compact, and let S and T be commuting continuous mappings of K into K such that

(3.6.1) S and T have *dimjodis*, and

(3.6.2) $f = ST = TS$ is bounded.

Then for each x in K, some subsequence of the sequence $\{f^n(x)\}$ converges to a point z in K and $Sz = Tz$.

COROLLARY 3.7. Let $K \sqsubseteq E$ be nonempty sequentially compact, and let S and T be commuting nonexpansive mappings of K into K having *dimjodis*. Then for each x in K some subsequence of the sequence $\{f^n(x)\}$ where $f = ST = TS$, converges to a point z in K and $Sz = Tz$.

Theorm 3.4 and its corollaries generalize the corresponding results of Tan [7] and Mishra [5].

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PLANT-PARASITIC AND SOIL NEMATODES INFESTING LEGUMINOUS CROPS IN RISHIKESH AND HARIDWAR REGIONS (U.P.)

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ABSTRACT

A survey of soil samples from the roots of Leguminous crops collected from ten localities in and around Rishikesh and Haridwar (U.P.) yielded 31 species of plant-parasitic and soil nematodes. These include mainly ectoparasites, predators and free-living forms, only two endoparasitic nematodes viz. *Rotylenchulus reniformis* and *Meloidogyne javanica* were recovered. *Helicotylenchus dihystra* and *Hopolaimus indus* are the most frequent nematods found in this region.

Key words : Leguminous crops, plant-parasitic and soil nematodes.

INTRODUCTION

Leguminous crops like *Phaseolus mungo*, *P. radiata*, *P. acontifolius*, *Cajanus cajan*, *Pisum sativum* etc. are widely grown in west Uttar Pradesh. Beside serving as fodder to animals these are also important, for Nitrogen fixation. Das [1] reported several nematodes associated with leguminous crops for the first time in India. Later on many nematodes including some endoparasites viz., *Heterodera avenae*, *H. cajani*, *H. trifolli*, *Meloidogyne javanica*, *M. incognita*, *Rotylenchulus reniformis*, *Radopholus similis* have been reported infesting

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leguminous crops in several parts of India by Seshadri & Shivkumar [11], Prasad *et al.*[9], [2], Koshy [5], Koshy and Sosamma [6], Janarthanan *et al.*[3], Sitaramiah *et al.* (12), Mulk & Jairajpuri [7,8], Kaushik & Bajaj [4], Reddy [10], A world list of Nematodes pathogens associated with chickpea, ground nut, pearl millect, pigeonpea and Sorghum was provided by Sharma [13]. But Rishikesh and Haridwar region has not yet been surveyed by any worker for the recovery of nematodes associated with leguminous crops. Therefore, an attempt has been made to recover all nematode associated with leguminous crops in this region.

MATERIALS AND METHODS

The soil samples were processed by Cobb's Sieving and decantation techniques. Nematodes were isolated with the help of Baermann's funnel. The nematodes, thus recovered, were fixed with hot 4% formaline. Mounting of nematodes was done in anhydrous glycerine after slow dehydration. DeMan's formula was used for denoting body dimension. Ten localities were chosen to collect the soil samples around the roots of different leguminous crops.

RESULT AND DISCUSSION

A survey of soil samples during 1989-90 around the roots of different leguminous crops yielded as many as 31 species of plant-parasitic and soil nematodes (Table-1). All these nematodes belong to four orders of Nematoda viz., Tylenchida, Dorylaimida, Mononchida and Rhabditida. The nematodes belonging to Tylenchida are ecto -and endoparasites. Dorylaimida includes free-living as well as ecto-parasites. Mononchids and Rhabditids are exclusively predatory and saprophagous respectively. Some nematodes have been studied in detail and are as follows:

1. *Tylenchorhynchus vulgaris* Upadhyay, Swarup and Sethi, 1972:

(5 ♀♀) L = 0.62 - 0.78 mm; a = 24-26, b = 4.1 - 4.4, c = 19-21, v = 53-55, Spear = 22-23 µm; nervering=105-110 um; excretory pore=115-128 µm.

2. *Helicotylenchus dihystra* (Cobb, 1912) Sher, 1963

(7 ♀♀) L = 0.48-0.54 mm; a = 27-30; b = 3.5-4.0; c = 35-38; v = 48-50; Spear = 23-25 µm; excretory pore = 105-145 um.

3. *Hopolaimus indicus* Sher, 1963

(10 ♀♀) L = 0.90-1.1 mm; a = 23-25, b = 4.3-4.6; c = 21-24; v = 54-57; Spear = 33-37 um; Nerve ring = 90-105 um; excretory pore = 85-98 µm.

(5 ♂♂) L = 0.82-0.90 mm; a = 21-23; b = 4.4-4.6 c=24-28; Spicules=35-40 um; Bursa = 100-115 µm.

4. *Rotylenchus* sp.

(5 ♀♀) L=0.87-0.95 mm; a=30-37; a=30-33; b=6.0-6.5; c=22-24; v=65-67; Spear=27-30 um; Nerve ring= 95-110 um; excretory pore=117-120 µm.

5. *Longidorus* sp.

(5 ♀♀) L=3.6-4.5 mm; a=135-148; b=11-14; c=55-62; v=51-54; Odontostyle=73-75 µm; Odontophore= 56-65 µm., Guiding ring = 23-27 µm; pre rectum = 230-245 µm, Basal Bulb=53-60 x 20-22 µm,

6. *Xiphinema* sp.

(2 ♀♀) L=1.30-1.32 mm; a=44-45; b=9-10; c=17;
V=45-46; Odontostyle=65 µm; Odontophore=73 µm

7. *Actinolaimus* sp.

(10 ♀♀) L=1.75-1.89 mm; a=51-55; b=6.7-7.2; c=6-7;
V=52-55; Odontostyle=18-20 µm; Odontophore=
21-24 µm.

8. *Thornenema* sp.

(5 ♀♀) L=1.30-1.36 mm; a=35-38; b=4.4-4.6; c=5.3-5.6;
V=38-40; Odontostyle=11-13 µm; Odontophore=
15-20 µm.

9. *Mylonchulus minor* (Cobb, 1893) Andrassy, 1958;

(10 ♀♀) L=1.05-1.15 mm; a=23-25; b=3.2-3.5; c=30-32;
V=57-58; Buccal cavity = 29-35 x 14-17 µm;
tail = 34-35 µm.

10. *Iotonchus indicus* Jairajpuri, 1969;

(9 ♀♀) L=1.78-1.90 mm; a=31-35; b=4.4-4.7; c=8.1-8.6;
V=57-58; Buccal cavity = 43-47 x 23-25 µm.

From the present study it is clear that large number of nematodes are associated with the leguminous crops in this region. *Helicotylenchus dihystra*, *Hoplolaimus indicus*, *Tylenchorhynchus -vulgaris*, *Eudorylaimus* sp., *Longidorus* sp. and mylonechulus minor were most abundant while *Hirschmanıella*, *Aphelenchus*, *Scutellonema*, *Paratrichodorus* and *Rotylenchulus* were least encountered here.

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TABLE-1

LIST OF NEMATODES RECORDED FROM DIFFERENT LOCALITIES & DIFFERENT SAMPLES (Numbrs)

	NAMES OF NEMATODES	NUMBER OF SAMPLES
1.	*Longidorus* sp.	1, 2, 5, 6, 7, 9
2.	*Hoplolaimus indicus*	1, 2, 3, 4, 8, 9
3.	*Helicotylenchus dihystra*	1, 2, 3, 6, 7, 8, 9, 10
4.	*Rotylenchus* sp.	6, 8, 9
5.	*Apercelaimus* sp.	1, 2
6.	*Hirschmanniella mucronata*	1
7.	*Pratylenchus* sp.	8
8.	*Rotylenchulus reniformis* sp.	3
9.	*Eudorylaimus* sp.	1, 2, 3, 4, 5, 7, 9
10.	*Dorylaimus stagnalis*	4, 5, 6, 7
11.	*Xiphinema* sp.	3
12.	*Prodorylaimus* sp	4, 6, 9
13.	*Tylenchorhynchus vulgaris*	1, 2, 3, 4, 6, 8, 10
14.	*Thornonema* sp.	3, 4, 6, 7, 8
15.	*Scutellonema* sp.	4, 2
16.	*Mononchus agnaticus*	4, 6
17.	*Amphidelus* sp.	5, 9
18.	*Mylonchulus minor*	5, 6, 7, 8, 9, 10
19.	*Discolaimus brevis*	4, 6, 9, 10
20.	*Aphelenchus avenae*	7
21.	*Para-phelenchus* sp.	7
22.	*Paratrichodorus mirzai*	7
23.	*Iotonchus indicus*	3, 7, 8, 10
24.	*Paratylenchus* sp.	8
25.	*Meloidogyne javanica*	8, 9
26.	*Aphelenchoides andrassyi*	8
27.	*Proleptonchus* sp.	9
28.	*Actinolaimus* sp.	2, 3, 5, 7, 8
29.	*Chronogaster* sp.	2, 3
30.	*Plectus* sp.	4
31.	*Rhabditis* sp.	8

Table-1 Contd.

LOCALITIES : SOIL AROUND THE ROOTS OF :

1. Pigeon Pea (*Cajanus cajan*) from Ram Nagar, Rishikesh, Dehradun.

2. Pigeon Pea (*Cajanus cajan*) from Geeta Nagar, Rishikesh, Dehradun.

3. Dhaicha (*Crotoloria junicea* L.) from Bhadra Bad, Haridwar.

4. Urd (*Phaseolus radiata*) from Sikenderpur, Haridwar.

5. Pigeon Pea (*Cajanus cajan*) from Dhanorie, Haridwar.

6. Bean Plant (*Dolichos lab-lab*) from Shyampur, Haridwar.

7. Chick Pea (*Cicer aretenum*) or Gram from Ganga Nagar, Rishikesh, Dehradun.

8. Matar (*Pisum sativum*) from I.D.P.L., Virbhadra, Rishikesh.

9. Chick Pea (*Cicer aretenum*) or Gram from Gomani Wala, Shyampur, Rishikesh, Dehradun.

10. Master (*Pisum sativum*) from Dhalwala, Rishikesh, Dehradun.

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बहुमानी संकारकों के अनुक्रमों का अभिसरण

श्याम लाल सिंह* उमेश चन्द्र गैरोला* एवं स्वामी नाथ मिश्रा**

(प्राप्त 6.12.1990)

सारांश

इस प्रपत्र में दो बहुमानी प्रतिचित्रणों के लिए इसिकावा पुनरावृत्तिक विधि के अधीन दो स्थिर बिन्दु प्रमेय स्थापित किये गये हैं, जो नैम्पल्ली-सिंह एवं सिंह व अन्य के परिणामों को विस्तारित करते हैं ·

AMS (MOS) Subject Classifications (1980, Revision 1985). 54H25, 47H10.

ABSTRACT

CONVERGENCE OF SEQUENCES OF MULTIVALUED OPERATORS

S.L. Singh*, U.C. Guirola* and S.N. Mishra**

The notion of Ishikawa iterates is extended to a pair of multivalued maps and it is shown hat under certain contractive conditions the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps.

भूमिका एवं प्रारंभिकी

रोअडेस् [10], हिक्स-कुबिसेक [3], तथा अन्य कई गणितज्ञों (देखें, उदाहरणार्थ [9], [10], [12]) ने दिखाया है कि कोई प्रतिचित्रण T जो किसी विशेष संकुचित शर्त को सन्तुष्ट करता है उसके स्थिर बिन्दु पर वह मान पुनरावृत्तिक (देखें [2], [14]) अनुक्रम अभिसरित होता है जो स्वयं अभिसारी हो, दूसरी ओर नैम्पल्ली-सिंह [9] ने व्यापक संकुचन शर्त के साथ यह देखा कि कोई अभिसारी इशिकावा [4] पुनरावृत्तिक अनुक्रम उसी बिन्दु पर अभिसरित होता है यदि वह बिन्दु प्रतिचित्रण T का स्थिर बिन्दु हो जब-कि नायडू-प्रसाद [8] ने [9] में प्रदत्त कुछ परिणामों का अध्ययन इशिकावा पुनरावृत्तिक अनुक्रम के अधीन एक युग्म प्रतिचित्रण के लिये किया हाल ही में कुहफिटिंग [6] ने मान पुनरावृत्तिक विधि हेतु क्रेस्नोसेस्कोज [5] की पुनरावृत्तिक विधि का विस्तार बहुमानी प्रतिचित्रणों के लिये किया इन सभी धारणाओं से प्रेरणा लेकर सिह [15] ने बहुमानी प्रतिचित्रणों के लिये इशिकावा पुनरावृत्तिक विधि को पारिभषित किया तथा इसके लिये स्थिर बिन्दुओं का सन्निकटन प्राप्त किया।

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प्रस्तुत प्रपत्र में एक युग्म बहुमानी प्रतिचित्रणों के लिए इशिकावा पुनरावृत्तिक विधि के अधीन दो प्रमेयों को सिद्ध कर रहे हैं. प्राप्त परिणाम सिंह [15] का व्यापकीकरण करते हैं. साथ ही रोअडेस् [10], हिक्स - कुबिसेक [3], नैम्पल्ली - सिंह [9], नायडू-प्रसाद [8] तथा कुहाफिटिग [6] द्वारा एक मानी एवं बहुमानी प्रतिचित्रणों के लिए स्थापित परिणाम विशिष्ट दशाओं के साथ समुचित रूप से प्राप्त किये जा सकते हैं .

सामान्यतया नाडलर [7] एवं सिंह [15] में प्रयुक्त संकेताक्षरों का अनुसरण किया गया है.

मान लें $(X, || \, ||)$ एक मानकित रैखिक समष्टि है, $CL(X)$ समष्टि X के अरिक्त संवृत उपसमुच्चयों का संग्रह है तथा $(CL(X), H)$ व्यापकीकृत हाउसडार्फ दूरीक समष्टि है.

निम्न प्रमेयिका सुज्ञात है (अधिक जानकारी के लिए देखें रूस [13]).

प्रमेयिका . मान लें $q > 1$ एवं $A, B \in CL(X)$. तब A के प्रत्येक अवयव x के लिए B में एक बिन्दु y का अस्तित्व इस प्रकार प्राप्त होता है कि

$$d(x,y) \leq qH(A,B).$$

हम X के संवृत अवमुख उपसमुच्चय C से $CL(C)$ पर पारिभाषित प्रतिचित्रणों S, T के लिए [15] में चर्चित निम्न पुनरावृत्तिक विधि का अनुसरण करेंगें .

(1) $x_o \in C$;

(2) $y_n = b_n p_n + (1-b_n) x_n, n \geq 0, \; p_n \in Sx_n$,

(3) $x_{n+1} = (1-a_n) x_n + a_n q_n, \; n \geq 0$;

उक्त प्रमेयिका के अनुसार $q_n \in Ty_n$ इस प्रकार है कि

(4) $||p_n - q_n|| = k^{-h} H(Sx_n, Ty_n)$, जहां $h, k \in (0,1)$;

(5) सभी n के लिए $0 \leq a_n, b_n \leq 1$;

(6) सीमा $a_n \geq 0$.

ध्यान देने योग्य है कि यदि $T: C \rightarrow C$ तो शर्त (4) आसानी से सन्तुष्ट हो जाती है क्योंकि $k^{-h} > 1$.

परिणाम

प्रमेय 1. मान लें C मानकित रैखिक समष्टि X का संवृत अवमुख उपसमुच्चय है तथा $S, T: C \rightarrow CL(C)$. पुनः मान लें कि पूर्व पारिभाषित अनुक्रम $\{x_n\}$ बिन्दु z पर अभिसरित होता है. यदि C के सभी अवयवों x, y के लिए S, T शर्त

$$(7)\qquad H(Sx,\ Ty) \le \text{अधिकतम} \{ ||x-y||,\ D(x,Sx), D(y,Ty), [D(y,Sx)+D(x,Ty)] \}$$

को संतुष्ट करे, तब $z \in Sz \cap Tz$.

उपपत्ति. क्योंकि $x_n \to z$ और $\{a_n\}$ शून्य से उपरि परिबद्ध है, अतः

$$||x_{n+1}-x_n|| = a_n||x_n-q_n||$$

एवं

$$||q_n-z|| \le ||z-x_n|| + ||x_n-q_n||$$

अर्थात् $||x_n-q_n||$ एवं $||q_n-z||$ शून्य पर अभिसरित होते हैं. पुनः त्रिभु असमिका एवं (2) से

$$||x_n-y_n|| = b_n||x_n-p_n|| \le ||x_n-p_n||$$

$$||x_n-p_n|| \le ||x_n-q_n|| + ||q_n-p_n||$$

$$\begin{aligned} ||y_n-q_n|| &\le ||y_n-x_n|| + ||x_n-q_n|| \\ &\le ||x_n-p_n|| + ||x_n-q_n|| \\ &\le ||x_n-q_n|| + ||p_n-q_n|| + ||x_n-q_n|| \\ &\le 2||x_n-q_n|| + ||p_n-q_n|| \end{aligned}$$

एवं

$$||y_n-p_n|| = (1-b_n)\ ||x_n-p_n|| \le ||x_n-p_n||.$$

अतः

$$||p_n-q_n|| \le k^{-h}H(Sx_n,Ty_n)$$

$$\le k^{-h} \text{ अधिकतम } \{ ||x_n-y_n||,\ D(x_n,Sx_n),\ D(y_n,Ty_n), [D(x_n,Ty_n)+D(y_n,Sx_n)] \}$$

$\leq k'$ अधिकतम $\{ ||x_n-p_n||, ||y_n-q_n||, ||x_n-q_n||$
$+ ||y_n-p_n||],$

जहाँ $k^{1-h} = k'$. इसलिए

$$||p_n-q_n|| \leq \frac{2k'}{1-k'} ||x_n-q_n||.$$

इस प्रकार $n \to \infty$ लेने पर

$$||p_n-q_n|| \to 0, \ ||x_n-p_n|| \to 0,$$

तथा

$$||p_n-z|| \to 0, \ ||y_n-q_n|| \to 0$$

एवं (2) से $||y_n-z|| \to 0$.

शर्त (7) द्वारा,
$D(z,sz)$

$$\leq ||z-q_n||+D(q_n,Sz)$$

$$\leq ||z-q_n||+H(Ty_n,Sz)$$

$$\leq ||z-q_n||+k \text{ अधिकतम}\{ ||y_n-z||, D(y_n,Ty_n),$$

$$D(z,Sz), \ [D(y_n,Sz)+D(z,Ty_n)\}$$

$$\leq ||z-q_n||+k \text{ अधिकतम}\{ ||y_n-z||, ||y_n-q_n||,$$

$$D(z,Sz), ||y_n-z|| + D(z,Sz) + ||z-q_n||]\}.$$

स्पष्टतः $n \to 0$ लेने पर, $z \in Sz$. इसी तरह $z \in Tz$.

अब प्रतिचित्रणों $S,T:C \to CL(C)$ के लिए निम्न शर्तों पर विचार करें·

(8a) $H(x,Sx)+H(y,Ty) \le a\|x-y\|,\ 1 \le a < 2;$

(8b) $H(x,Sx) + H(y,Ty)$

$\le b[D(x,Ty) + D(y,Sx) + \|x-y\|]$, जहाँ $1/2 \le b < 2/3$,

(8c) $H(x,Sx) + H(y,Ty) + H(Sx,Ty)$

$\le c[D(x,Ty) + D(y,Sx)],\ 1 \le c < 3/2$,

(8d) $H(Sx,Ty) \le k$ अधिकतम $\{\|x-y\|, D(x,Sx)$,

$D(y,Ty),\ 1/2[D(x,Ty) + D(y,Sx))\},\ 0 \le k < 1$

प्रमेय 2. मान लें C मानकित रैखिक समष्टि X का एक संवृत अवमुख उपसमुच्चय है एवं पूर्व पारिभाषित अनुक्रम $\{x_n\}$ किसी बिन्दु $z \in C$ पर अभिसरित होता है. यदि C के प्रत्येक अवयव x,y के लिए प्रतिचित्रण S,T उक्त शर्तों (8a)-(8d) में से कम से कम एक शर्त को सन्तुष्ट करे तथा (8e) भी सन्तुष्ट हो तब $z \in S_z \cap T_z$, जहाँ

(8e) $k^{-h} \le 2$ या सीमा $b_n > \dfrac{(c-1-k^h)}{(c+1)}$.

उपपत्ति. हम किरिक [1] के निम्न सरल कथन का कई बार प्रयोग करेंगें,

किसी $x \in C$ एवं $y \in B$ के लिए

$$\|x-y\| = H(x,B),\ B \in CL(C).$$

चूँकि $p_n \in Sx_n$ तथा $q_n \in Ty_n$,

$$2\|p_n-q_n\|$$

$$\le (\|x_n-p_n\| + \|y_n-q_n\|) + \|x_n-q_n\| + \|y_n-p_n\|$$

$$\le H(x_n,Sx_n) + H(y_n,Ty_n) + \|x_n-q_n\| + \|y_n-p_n\|.$$

इनसे जैसा कि प्रमेय 1 की उपपत्ति में देखा गया, निम्न सरलता से प्राप्त होते हैं :

$$||x_n - q_n|| \to 0 \qquad ||q_n - z|| \to 0.$$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8a) सन्तुष्ट हो तब

$$2\,||p_n - q_n||$$

$$\le a||x_n - y_n|| + ||x_n - q_n| + ||y_n - p_n||$$

$$\le ab_n(||x_n - q_n|| + ||q_n - p_n||) + ||x_n - q_n||$$

$$+ (1 - b_n)\,(||x_n - q_n|| + ||q_n - p_n||)$$

अर्थात्

$$||p_n - q_n||$$

$$\le t||x_n - q_n||, \text{ जहाँ } \quad t = (2 + (a-1)\,b_n)/(1-(a-1)b_n).$$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8b) सन्तुष्ट हो, तब

$$2||p_n - q_n||$$

$$\le b(\,D(x_n, Ty_n) + D(y_n, Sx_n) + ||x_n - y_n|| + (||x_n - q_n||$$

$$+ ||y_n - p_n||)$$

$$\le (b+1)\,(||x_n - q_n|| + ||y_n - p_n||) + b||x_n - y_n||$$

अर्थात्

$$||p_n - q_n||$$

$$\le t_2\,||x_n - q_n||, \qquad t_2 = (2 + 2b - b_n)/(1 - b + b_n).$$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8c) सन्तुष्ट हो, तब

$||p_n-q_n|| \le t\, H(Sx_n, Ty_n)$, जहाँ $t = k^{-h}$.

अतः

$$(2t + 1)\, ||p_n-q_n|| = t\, ||p_n-q_n|| + t\, ||p_n-q_n||$$

$$+ ||p_n-q_n|| \le t(||p_n-y_n|| + ||y_n-q_n||)$$

$$+ t(||x_n-p_n|| + ||x_n-q_n|| + tH(Sx_n, Ty_n)$$

$$\le t\, H(y_n, Ty_n) + tH(x_n, Sx_n) + tH(Sx_n, Ty_n)$$

$$+t(||x_n - q_n|| + ||y_n-p_n||)$$

$$\le ct\, D(x_n, Ty_n) + D(y_n, Sx_n) + t\, (||x_n-q_n|| + ||y_n-p_n||)$$

$$\le t(c+1)\, (||x_n-q_n|| + ||y_n-p_n||)$$

अथवा

$$||p_n-q_n|| \le t_3 ||x_n-q_n||,$$

जहाँ $t_3 = t(c+1)\,(2-b_n)/(1+t(1-c(1-b_n)+b_n))$.

अन्ततः यदि $x = x_n$ एवं $y = y_n$ के लिए (8d) सन्तुष्ट हो, तब

$$||p_n-q_n|| \le$$

$$tk \text{ अधिकतम } \{||x_n-y_n||,\ D(x_n, Sx_n),\ D(y_n, Ty_n),$$
$$1/2[D(x_n, Ty_n) + D(y_n, Sx_n)]\}$$

$$\le tk(2||x_n-q_n|| + ||q_n-p_n||).$$

अर्थात्

$$(10c)\ (2-c)\ H(z,Sz)$$

$$\leq (c+1)\ ||z-y_n|| + c||z-q_n||.$$

अतत: (8d) से

$$D(z, Sz) \leq ||z-q_n|| + H(Ty_n, Sz)$$

$$\leq ||z-q_n|| + k \text{ अधिकतम } \{||y_n-z||,\ D(y_n, Ty_n),$$

$$D(z,Sz),\ 1/2(D(y_n,z) + D(z,Ty_n))\}$$

$$(10d) \leq ||z-q_n|| + k \text{ अधिकतम } \{\ ||y_n-z||,$$

$$||y_n-q_n||,\ D(z, Sz),\ 1/2(||y_n-z|| + D\ (z,Sz)$$

$$+\ ||z-q_n||)\}.$$

अतः (10a) - (10c) में सीमा $n \to \infty$ लेने पर

$H(z,Sz) = 0$ एवं (10d) से $D(z,Sz) = 0$ अर्थात् $z \in Sz$.

इसी प्रकार हम सिद्ध कर सकते हैं कि $z \in Tz$.

टिप्पणी 1. प्रमेय 1-2 में $S = T$ लेने पर सिंह [15] के परिणाम उपप्रमेय के रूप में प्राप्त हो जाते हैं.

टिप्पणी 2. प्रमेय 1 में $S=T: C \to C$ लेने पर नैम्पल्ली-सिंह [9] की प्रमेय 1.2 को उपप्रमेय के रूप में प्राप्त किया जा सकता है.

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फार्म—४

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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३–मुद्रक का नाम : राजेन्द्र कुमार मदान
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व पता : गुरुद्वारा रोड़, ज्वालापुर, हरिद्वार
☎ : ४५०३३१
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राष्ट्रीयता : भारतीय
व पता : कुलसचिव गुरुकुल कांगड़ी विश्वविद्यालय
हरिद्वार–२४९४०४
५–प्रधान सम्पादक : डॉ० एस० एल० सिंह
राष्ट्रीयता : भारतीय
व पता : गणित विभाग, गुरुकुल कांगड़ी विश्वविद्यालय
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CONTENTS

BIOLOGY

103 I. P. Joshi and V. Shanker : Studies on Conservation Value of Common Plants of Shivaliks.
153 D R. Khanna and S. P. Badola : Ichthyofanda of the River Ganga at the Foothills of Garhwal Himalaya.
179 Subhash Chand Chauhan and Maqsood Ahmed : Plant-Parasitic and Soil Nematodes Infesting Leguminous Crops in Rishikesh and Haridwar Regions (U.P.).

MATHEMATICS

1 Shyam L. Singh : Extensions of the Caristi–Kird Fixed Point Theorem.
13 V. K. Katiyar, P. Goel and A. Sahai : Flow of Blood Through a Locally Constricted Tube – A Methematical Model and its Statistical Analysis.
35 O P. Narayan and D. P. Verma : Euler Maclaurin Formula for Two Dimensions.
43 Pramod Kumar Mittal and R. Mall : On Some Self–Superposable Fluid Motions in Prolate Spheroidal Ducts.
59 P K. Chaudhari and Sabyasachi Sen : Effect of Non-Homogeneity of Critical Load in Buckling of a Bar Under Distributed Axial Load.
81 Subrata Datta : Note on the Deflection in a Simply Supported Timoshenko Beam of Nonhomogeneous Material.
95 K. P. R. Rao : Common Fixed Point Theorems in Matric Linear and Uniform Spaces.
111 P. R. Subramanian : Significance of the Constant Term of a Hermite Polynomial of Even Degree.
121 W B. Vasantha : A Note on the Modular Group Ring of the Symmetric Group S_n
125 Subhash Chandra Nandy : An Alternative Approach to Tollmien's Energy Spectra Derived from Heisenberg's Equation of Turbulent Flow.
137 B G. Pachpatte : On the Stability of a Stochastic Integrodfferential Equations in a Banach Space.
163 S. N. Mishra and S. L. Singh : Coincidence Theorems in Linear Topological Spaces.
187 श्यामलाल सिह, उमेशचन्द गैरोला एवं स्वामीनाथ मिश्रा : बहुमानी संकारकों के अनुक्रमों का अभिसरण ।

PHYSICS

25 B. Kumar, H. C. Verma and D. P. Verma : Lattice Sum of Electric Field Gradients in Orthorhombic Crystal.
69 P. P Pathak : Geophysical Prospecting with Lightning Radiation.

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'CHEMICAL INVESTIGATIONS OF SOME INDIGENOUS BASIL OILS'

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ABSTRACT

Chemical composition of essential oil of *Ocimum sanctum* Linn distributed in and around Rishikesh and from the plants raised at Haldwani from the seeds of the plants distributed at Rishikesh, have been studied by derivatisation, TLC, GLC, PMR, MS and IR. In all twenty compounds have been identified as against nine compounds reported in literature, Principal components are : eugenol, methyl eugenol, and methyl chavicol. The Rishikesh oil sample containing 66.5 percent eugenol can serve as a substitute for oil of cloves.

Keywords and Phrases : Essential oil, *Ocimum sanctum*, Chemistry-subject classification:Pharmaceutical analysis Essential Oils - 62.

INTRODUCTION

Ocimum sanctum, belonging to N.O. Labiateae, is widely distributed in the warmer parts of the world. In India it is commonly known as 'Sri tulsi' and 'Krishna tulsi' and holds promise due to its high eugenol content. It is held sacred and the plant and its parts are credited possessing numerous curative effects probably not shared by many other plants.

Essential oil from *O. sanctum* has not been thoroughly investigated. Only a few constituents have been reported viz. linalool, geraniol, cirtronellol, cineole and methyl chavicol in the oil of plants raised at Phillipines by Guenther [5]; Cineole, eugenol and methyl chavicol by Gildmeister et al. [3]; eugenol, methyl chavicol, carveol and caryophyllene by Datt et al. [2]; eugenol, methyl eugenol and caryophyllene by Lawrance et al. [12]; eugenol, methyl chavicol, linalool and 1:8 cineole by Satti et al. [15]; Gibberellic acid (GA_3) treatment of the crop of *O. sanctum* has been reported increasing the essential oil as well as eugenol content [6]; Harvesting the crop at full bloom

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possessed maximum eugenol content [7] [1] [16] [18] and that the oil possessed larvicidal, synergistic and antibacterial/ antifungal properties [1] [17]. In India and Pakistan the oil is used as a substitute for oil of cloves [8].

Essential oils were extracted, separately, from the plants of *O. sanctum* distributed at Rishikesh and from the plants raised at Haldwani (another place on the foot hills of Himalayas) from the seeds of Rishikesh plants. Chemical composition of the two oil samples were studied in order to know the difference, if any, in their composition due to change of place of distribution/cultivation.

EXPERIMENTAL :

Essential oil was obtained by steam distillation of the fresh plant material. The oil was found mostly concentrated in leaves and flowers. Stalks contained negligible oil. Oil content in leaves and flowers at various phases of plant development are given in table.1.

TABLE 1

OIL CONTENT IN LEAVES & FLOWERS AT DIFFERENT STAGES OF PLANT DEVELOPMENT AT RISHIKESH

Stage of growth	part of the plant	oil content (fresh weight basis)
Pre flowering (immature inflorescence)	leaves	0.65%
	flowers	0.60%
Flowering	leaves	0.58%
	flowers	0.41%
Post flowering	leaves	0.56%
	flowers	0.41%

Physico-chemical properties of the essential oil obtained from the plants (flowering stage) distributed at Rishikesh were:

Sp. gravity/20° = 0.9498 ; Ref. index/20° = 1.5184 ;
Sp. rotation 20° = - 6° 20' ; Acid value = 1.2 ; Ester value = 24.1 ;
Carbonyl value as $C_{10} H_{16}0$ (Oximation) = 1.5 ;
Phenolic content = 66.2 and solubility in 80% alcohol = 1:0.60.

Chemical composition of the essential oil was studied by the usual derivatization, chromatographic processes like TLC & GLC and IR, UV, PMR and Mass spectroscopy. 50 ml of the oil was shaken thoroughly with 5% aqueous sodium hydroxide followed by 1% sodium hydroxide. The oily layer was separated from the reaction mixture by extraction with petroleum ether (40-60°) and the non-phenolic constituents (16.8 ml) were obtained by distilling off the solvent from the ether extract. The aqueous sodium hydroxide extract was acidified with 20% phosphoric acid to Congo red point and extracted with ether. Ethereal extract was dried over anhydrous sodium sulphate and on distilling off the solvent 32.5 ml phenolic contents were obtained. Constituents present in phenolic and non-phenolic fractions were investigated separately.

PHENOLIC FRACTION :

TLC examination over silica gel G layers under different solvent systems and GLC indicated this fraction to be a single compound. The fraction possessed $d_4^{20} = 1.0616$; $n^{20}_D = 1.541$ and formed phenyl urethane m 97°, benzoate m 69°and tetrabromide m 118° [10] [9] [4]. These properties and IR spectra confirmed this fraction to be eugenol.

NON-PHENOLIC FRACTION :

This fraction was subjected to further separation -

a) Separation of alcohols :

The alcohols from the non-phenolic fraction (16.8 ml) were separated by refluxing the fraction with calculated amounts of 3:5 Dinitro benzoyl chloride and pyridine. The reaction mixture was cooled and water was added. The solid derivative was separated and dried.

TLC examination of the solid derivative over silica gel G coated plates, using benzene : petroleum ether (1:1) as solvent system, butter yellow as reference compound and ethanolic α - naphthyl amine solution (1%) as spray reagent showed the presence of linalool ($R_B = 1.75$), α- terpineol ($R_B = 1.4$) and citronellol ($R_B = 1.65$). Identity of these compounds was further confirmed by Co - TLC.

b) Alcohol free non-phenolic fraction :

The fraction (12.5ml), left after distilling off the pyridine under reduced pressure, was separated quantitatively into 'Hydrocarbon fraction' and

'Oxygenated fraction' by using a column packed with silicic acid, and n-hexane and ethyl acetate, respectively, as eluting agents [11].

Hydrocarbon fraction :

TLC examination of the fraction over silica gel G coated plates, using n-hexane as solvent system, limonene as reference compound, fluorescein-bromine as reagent for visual identification showed the presence of three compounds viz. β - elemene (R_L = 1.3), β - caryophyllene (R_L = 1.4) and terpinolene (R_L = 1.5). Identity of these compounds was confirmed by Co-TLC, and PMR/Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Hydrocarbon fraction' [13] [14] [19] [20].

Oxygenated fraction :

TLC examination of the fraction over silica gel G layers using petroleum ether - pyridine (95:5) as solvent system, thymol as reference compound, suitable spray reagents viz. $SbCl_5 : CCl_4$ in 1:4 for ethers and fluorescein - bromine for esters for visual identification showed the prescence of methyl eugenol (R_T = 2.3), methyl iso-eugenol (R_T= 2.2) and eugenol acetate (R_T = 2.5). Prescence of these compounds was confirmed by Co-TLC. Identity of the compounds was finally confirmed by PMR/ Mass spectroscopy of the compounds obtained from preparative chromatography of the 'Oxygenated fraction' [13] [14] [19] [20].

GAS CHROMATOGRAPHY OF THE OIL :

Chemical composition of the oil was also studied by Gas chromatography using Perkin Elmer Model 3920. In order to know the difference in chemical composition, the two oil samples viz. from the plants distributed at Rishikesh and from the plants raised at Haldwani, were analysed by Gas chromatography under similar parameters. Different constituents were identified by 'injection method' and comparison of retention time under the same operating parameters using pure compounds. The conditions maintained were :

Column	3.6m long, 2.0 mm I.D.,
Stationary phase	-UCCW 10% in chromosorb, 80-100 mesh.
Column temperature	70-200°C ; @ 2° per minute.
Chart speed	5 mm per minute.
Detector & injector	

temperature	......	250° C.
Attenuation	......	512 & 256.
Range	......	10
Sample size	......	4µl-6µl (1:30 solution in $CHCl_3$).

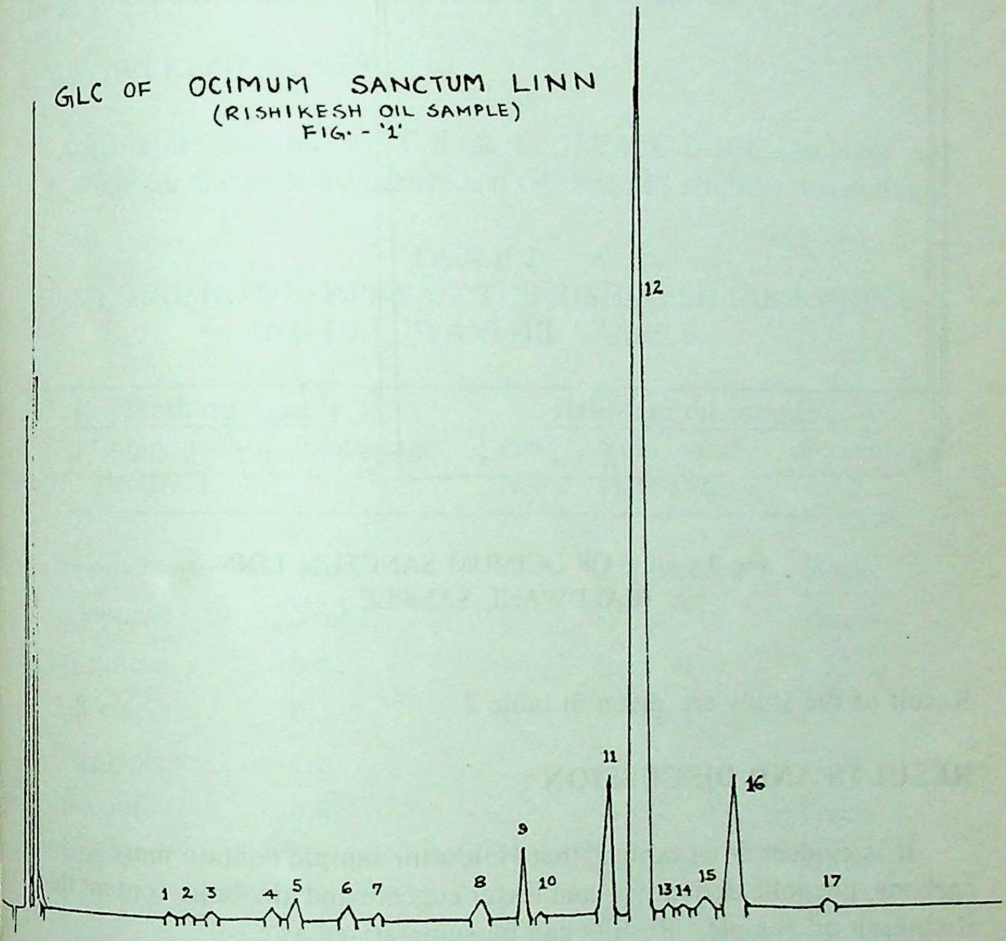

Fig 1 : GLC OF OCIMUM SANCTUM LINN
(RISHIKESH OIL SAMPLE)

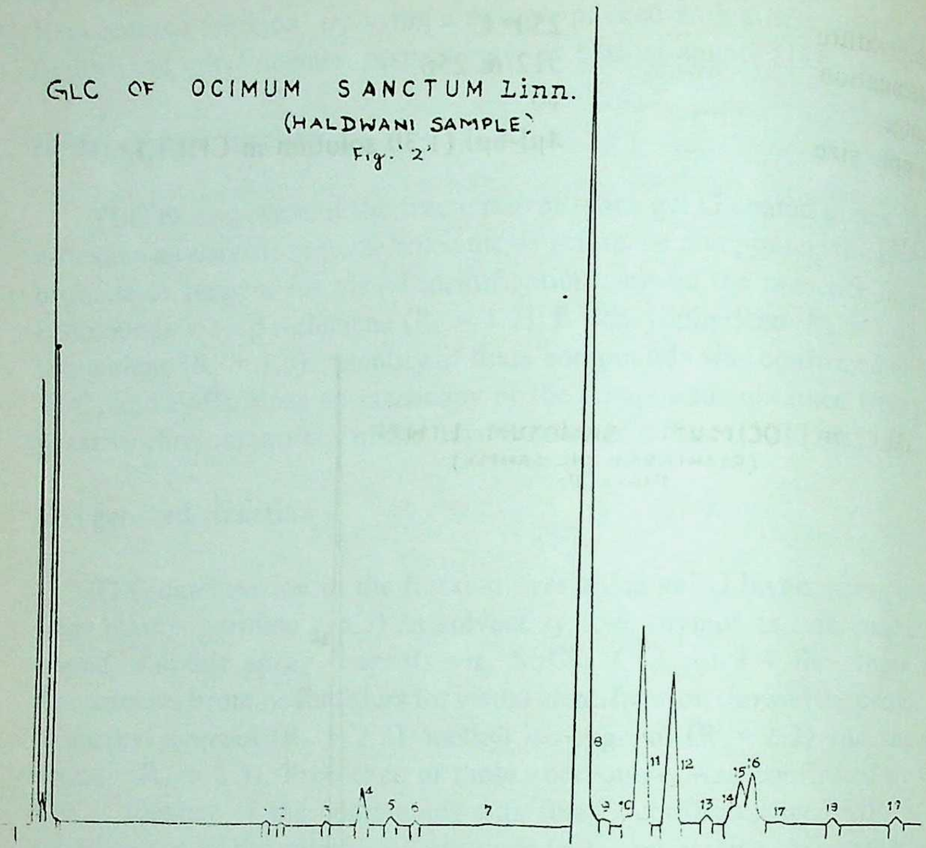

Fig 2 : GLC OF OCIMUM SANCTUM LINN
(HALDWANI SAMPLE)

Result of the study are given in table 2.

RESULTS AND DISCUSSION :

It is evident from table 2 that Haldwani sample contain more hydrocarbons, phenolic derivatives and lesser eugenol and alcoholic content than Rishikesh oil sample. Results can be summarised as :

	Haldwani sample	Rishikesh sample
Hydrocarbons	20.72%	12.85%
Phenolic derivatives	24.00%	0.6%
Eugenol	61.4%	66.5%
Alcoholic content	1.72%	7.3%

Difference in chemical composition of two oil samples may be attributed to the change of locality/place of cultivation of the plant. Further work is needed to ascertain the actual factors responsible for the difference. The authors have identified twenty compounds in the oil of *O. sanctum* as against nine compounds viz. linalool, geraniol, citronellol, cineole, methyl chavicol, eugenol, carveol, methyl eugenol and caryophyllene, reported as a result of work carried out by number of workers mentioned in literature [2] [3] [5] [12] [15]. Further, Rishikesh oil sample contanining 66.5 percent eugenol can serve commercially as a substitute for Oil of cloves.

ACKNOWLEDGEMENTS :

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TABLE 2
GAS CHROMATOGRAPHY OF THE RISHIKESH OIL SMAPLE AND HALDWANI OIL SAMPLE

Rishikesh oil sample			Haldwani oil sample		
Peak No.	Compound identified	Percentage	Peak No.	Compound identified	Percentage
1.	α- pinene	Trace	1.	1:8 cineole	Trace
2.	camphene	Trace	2.	-	Trace
3.	α- pinene	0.6	3.	linalool	0.12
4.	1:8 cineole	0.6	4.	borneol	1.6
5.	Terpinolene	1.0	5.	-	Trace
6.	linalool	0.8	6.	-	Trace
7.	camphor	0.1	7.	geraniol	Trace
8.	α - terpineol	1.2	8.	eugenol	61.4
9.	citronellol	4.9	9.	-	Trace
10.	geraniol	0.4	10.	-	Trace
11.	bornyl acetate	11.9	11.	methyl eugenol	20.15
12.	eugenol	66.5	12.	β-caryophyllene	17.4
13.	-	Trace	13.	humulene	0.68
14.	-	Trace	14.	methyl isoeugenol	1.65

15.	methyl eugenol	0.6	15.	aceto eugenol	2.2
16.	β-caryophyllene	11.1	16.	β-elemene	2.1
17.	β - elemene	0.15	17.	γ-elemene	Trace
18.	cadinene	0.52			
19.	-	0.56			

-denotes in this table the unidentified peaks.

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PHARMACOGNOSTIC STUDIES IN LEAF OF *LENOTIS NEPETAEFOLIA* L.

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ABSTRACT

The paper deals with Pharmacognosy of *Leonotis nepetaefolia* leaf including its morphological, anatomical, chemical constituents and powder analysis. Contrary to the earlier reports the leaves are amphistomatic, stomata are mostly diacytic, few isotricytic. Four types of trichomes viz. unicellular conical hairs, uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs have been recorded. The venation is Craspido-brochidodromous which is a new pattern. Midrib consists of one large semicircular shaped and petiole with five, one median large arch shaped and four laterals, two on each of the arms of median bundles. Powder analysis microscopically show fragments of epidermis, mesophyll, trichomes, tracheary elements. Positive tests for glycosides, alkaloids polyphenolases, flavones, juglones are recorded.

Keywords : *Leonotis nepetaefolia.* Pharmacognosy.

INTRODUCTION

The leaves of *Leonotis nepetaefolis* are curative and consider to be emmenagogue februfuge, depurative, narcotic, bitter and laxative: used in skin diseses, amenorrhoea and fevers [1,2,16]. The plant is medicinally used by tribe mundas of Chotanagpur. When a mother's breasts are well and milk does not pass through the nipples, the leaf, juices is also expressed and taken with lime juice and run as a febrifuge [9].

MATERIAL AND METHODS

Twigs of *Leonotis nepetaefolia* were collected from plants growing in Osmania University campus, Hyderabad, Andhra Pradesh. The methodology and terms used are after Prabhakar and Ramayya [18]. Leelavathi et al. [11] Prabhakar and Leelavathi [6]

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Abbreviations used in the text are : Ab: abaxial surface of the organ, Ad: adaxial surface of the organ, D: diameter, Dist: distribution, E.C.F.: epidermal cell frequency, L/W: length/width, S/F.: stomatal frequency.

OBSERVATIONS AND DISCUSSION

Vernacular names : BENGAL: RAJURCHEI; BOMBAY: Matijer, Matisul; BRAZIL; Cordao de frade; CEYLON: Kasitumpai; GUJARATI: Matizer, Matisul, HINDI: Bejurchei, MARATHI: Dipmal, Ekri, MUNDAR: Agaiajanum, Gharia, Matsuenggelsui, Singgelsui, PORTO RICO: Rascamono, SANTALI: Daredhompo, Jonumdhompo; SIMHALESE; Mahayakwanassa; TELUGU; Rana beri; Hanumantabira; Mulagolimedi.

Morphology: Erect. branched, pubescent, aromatic, herb, upto 2 m high, Root: Tap root Stem: branched, obtusely quandrangular with ridged angles, finely pubescent. Leaves : Simple, opposite-deccusate, ovate 10-13 x 7-9 cm, margin serrate, apex acute, pubescent. Petiole: Four to five cm. filifrm, pubescent. Inflorescence: Axillary, verticillaster, globose, about 6 cm in diameter. Flowers: Small, orange-scarlet, sessile, Bract : linear, 6 mm long strongly spinous, pointed, pubescent, Bracteoles: Two bract-like, 1-1.5 mm. Calyx : Somewhat bilabiate; tube 3 m long; lobes subequal, subulate, 2 mm long.. Corolla: Petals 5, bilipped; tube 9 mm, glabrous below, densely clothed above with scarlet orange hairs; upper lip 11 mm; lower lip 3 lobed 6 mm, lobes oblong. Stamens: Four didynamous, filaments subequal, 4-6 mm, anther 2-celled, cells divaricate, 1.25 mm. Ovary: Truncate, deeply 4-lobbed, pubescent, ovule solitary in each locule on axile placentation. Style: Filiform 1.5 cm glabrous. Stigma: Subequally bilobed. Fruit : Aggregate of 4 nutlets, nutlets oblong, 3-gonous, Flowering and fruiting : October to December.

Microscopic characters

Surface view of leaf : Anticlinally walls of laminar epidermal cells were described to be wavy to sinuate [5]. Presently the epidermal cells are polygonal anisodiametric and rarely polygonal linear. Anticlinal walls are sinuate sinuses being "V" shaped on abaxial but wavy to sinuate on adaxial sides. In none of the Lamiaceae taxa papillate epidermal cells were reported [5,10,15,19], however, presently papillated epidermal cells are observed on both abaxial and abaxial surfaces which is an important diagnostic charcter and has been reported only in few angiospermous taxa. Anticlinal walls are slightly thick and cytoplasmic contents are scanty. Epidermal cells are irregularly arranged and variously oriented Dist : All over the leaf lamina except on veins (Figs. 1&2).

Foliar costal cells and stomatal distribution patterns were considered to be of identification value [1]. The costal cells in the present taxon are mostly polygonal linear and a few rectangular linear. Anticlinal walls are straight and slightly thick. The costal cells are parellely oriented and irregularly arrange. Dist : On all grades of veins.

Type of the stomata in *Leonotis nepetaefolia* was reported to be diacytic [5,10,15,19,20]. However, presently the dominant diacytic stomata are found to be associated with few isotricytic stomata on abaxial side. Stomata are monocyclic. (Fig. 1&2). Shape of the guard cells are elliptic. Size of the stomata on adaxial and abxial surface range from 23-18-15 µm in (length) and 15-12-11 µm (width). Subsidiaries are mostly "a" type rarely of "f" type and are indistinct from epidermal cells (Fig. 1&2). Dist :The leaves were reported to be hypostomatic by Inamdar and Bhatt [5] however presently they are observed to be amphistomatic and distributed all over the leaf lamina except on veins of both the surfaces (Figs. 1&2).

In the past glandular (uni or bicelled head) and non glandularl (uniseriate and multicellular) trichomes were reported [5,10,15,19,20]. However there were no reports on presence of unicellular conical hairs which are presently observed. Besides other types like uniseriate conical hairs, uniseriate capitate hairs, uniseriate peltate hairs are also observed which are described below:

1. Unicellular conical hair : Foot : consisting of the basal end of the hair, indistinct, contents slightly dense; walls thin. Body : short or long, conical, apex acute, contents scanty; walls thin; surface smooth (Fig. 6).

2. Uniseriate conical hairs : Foot : One - celled, embedded or slightly projected above the epidermis, indistinct, contents scanty, walls thin. Body: Conical unicellular to uniseriate, 1 to 3 - celled in length; cells longer than broad contents scanty, walls thin, surface smooth (Fig. 7).

3. Uniseriate capitate hairs : Foot : One-celled, embedded, or slightly projected above the epidermis indistinct, usually broader than long; contents scanty; walls sthin Stalk : one to three celled; two tired; broader than long or squarish, contents scanty; walls thin surface smooth. Head : Eight celled, cells longer than broad; contents dense; walls thin surface smooth (Fig. 8).

4. Uniseriate peltate hairs : Foot : One-celled embedded or slightly projected above the epidermis, indistinct, usually broader than long; contents scanty; walls thin. Stalk : One to three-celled, two tired, broader than long or squarish, contents scanty; walls thin surface smooth, Head : Four or eight celled, cells longer than broad; contents dense; walls thin; surface smooth (Fig. 9).

Foliar venation as taxonomic tool has been since long time in use [16] which is equally important in Pharmacognostic studies [6-8,12]. Presently it is observed that most of the secondaries in *Leonotis* show branching. Further the basal secondaries terminate at margins (like in Craspidodromous pattern) and distal secondaries join the superadjacent secondaries forming loops (as in Brochidodromous pattern). This pattern of venation do not fit under any of the pattern described by Hickey [4] or Melville [14]. Hence presently described as a new pattern Craspido-brochidodromous which can be used as significant character for identification of the foliar drugs. (Fig. 3). Secondaries are six pairs, sub-opposite to opposite to alternate. The proximal loop forming branches joing the super adjacent secondaries at right angles to obtuse angle. Tertiaries are percurrent and their angle of origin is acute: right angle (AR). The relationship of the tertiaries to midvein is oblique to perpendicular and their course is retroflexed and predominantly opposite. Higher order is observed upto 5° vein.Areoles are polygonal, regular in orientation and 600/cm^2. Veinlets are simple and branched once, veinilets are 0-1/areole and 500/cm^2 (Fig. 5).

Sectional view of leaf : In transaction of leaf lamina is ribbed abaxially at primary, secondary and tertiary veins but adaxially inconspicuously ribbed only at primary veins (Fig. 3). Epidermal cells are circular to oval shaped, abaxial ones are smaller in size. (on lamina wings Ad: L/W=41-25-19/19-14-11; Ab:L/W = 30-21-15/15-13-11; midrib D: 19-15-11/16-28-8 µm) with thin cuticle and scanty contents. Guard cells with upper ledges. Palisade one layered. Cells are cylindricl and loosely arranged, (L/W: 79-57-49/23-15-11 µm). Near midvein and major lateral veins on both sides, hypodermis consists of 1-4 layered compactly arranged angular collenchyma (D=27-17-11 µm) and rest the ground tissue is parenchymatous. Parenchyma is 3 to 4 layers; cells polygonal to circular in shape(D-46-33-27 µm). Midvein consists of central large, semi-circular shaped single vascular bundle which is collateral and apericyclic. (Figs. 3). Tracheary elements are polygonal to circular (D=19-13-8 µm) more in midvein and secondary veins; few in others, arranged in rows. Secondary wall thickenings annular (free) helical (helices single or double). Perforation plate is simple.

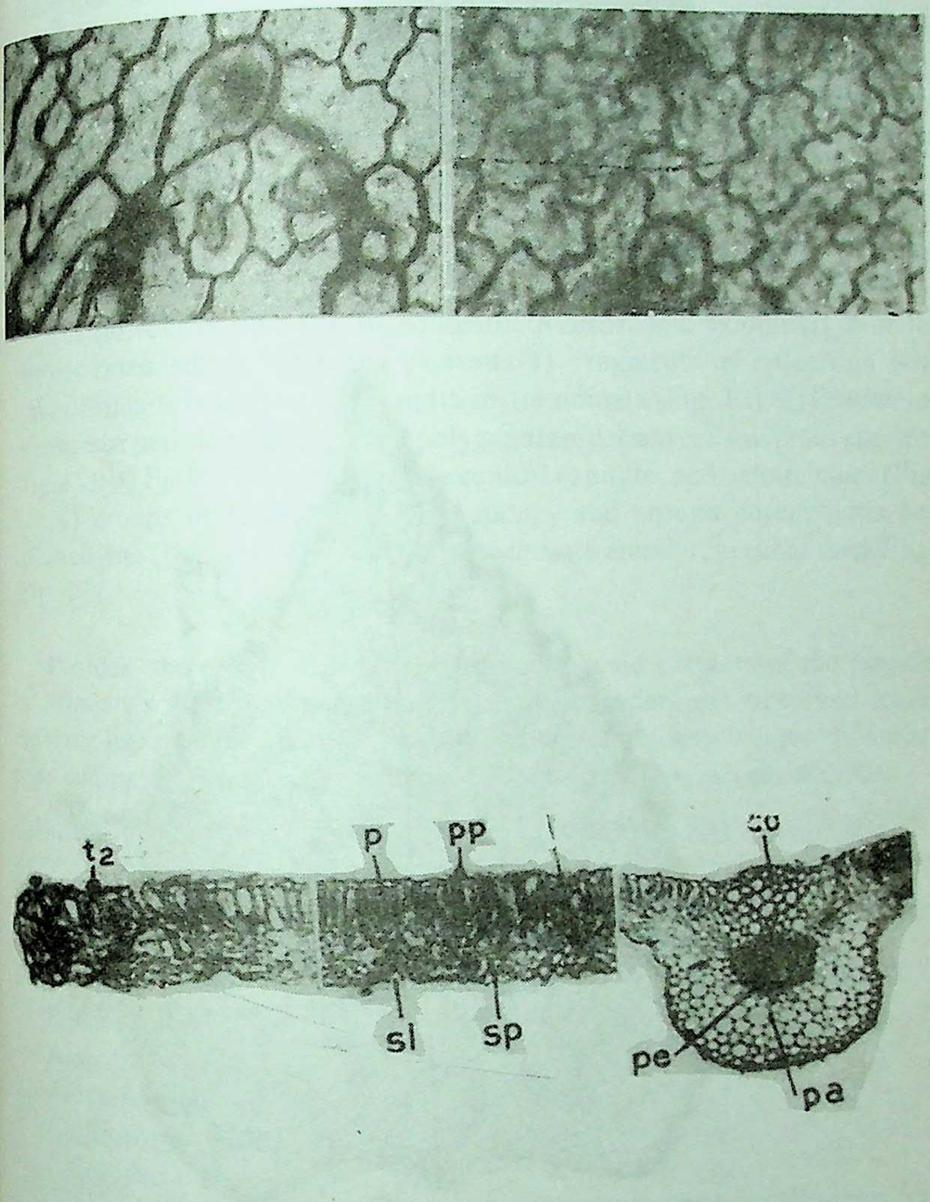
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Fig. 1-3

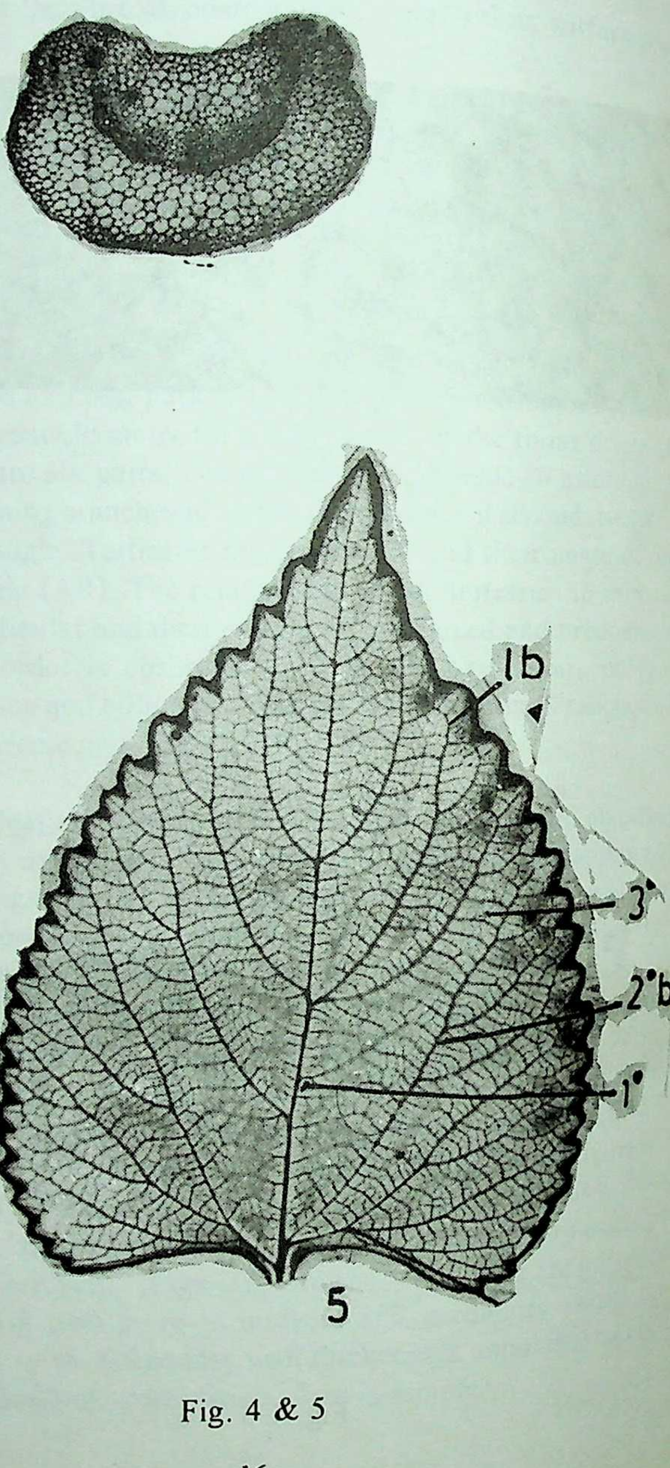

Fig. 4 & 5

In transection the petiole is circular and adaxially groved (Fig. 4). Epidermal cells are circular to oval in shape (D=15-13-11 μm). Ground tissue is heterogenous with 2 to 3 layered compaetly arranged collenchymatous hypodermal circular to polygonal.cells (19-17-15 μm) with scanty contents. Rest of the ground tissue is 5 to 8-layered parenchyma. Cells are polygonal (D=87-66-46 μm) with thin walls and scantly contents. Vascular bundles are 5; the median one is large, lunar shaped and the four small laterals, two on each of the arms of median bundle (Fig. 4). Tracheary elements are polygonal to circular (23-17-13 μm) arranged in rows. Secondary wall thickenings are annular (Free), helical (helices single or double) with simple perforation.

Powder Study

The powder is light brown, aromatic. Microscopic examination of the powder revealed the following elements 1) Fragments of epidermis with wavy or sinuate anticlinal walls and diacytic stomata (Fig. 10); 2) Fragments of costae or petiole showing small polygonal epidermal cells without stomata (Fig. 11); 3) Full or bits of uniseriate conical capitate, and peltate hairs (Fig. 12); 4) groups or isolated palisade, spongy and ground parenchyma and collenchyma (Fig. 13-16); 6) bits of vessels with annular, hellical thickeings (Fig. 17).

Besides the microscopic observations the cold extracts of the powder in different solvents and the moist and dry powders are observed under ordinary light and Ultraviolet light to record the colour changes. (Colours were compared with colour index [13] The details are given in table.

OBSERVATION OF LEAF UNDER ORDINARY AND U. V. LIGHT

Sl. No. Powder	Ordinary Light	U. V. light
1. Dried powder	CB	LB
2. Water extract	RM	DG
3. Residue	DP	GB
4. Alcholic extract	GY	GB
5. Alcholic residue	GY	DP
6. Chloroform extract	OB	GB
7. Chloroform residue	BB	OB

BB. Brown black; CB. clay brown; DG. dark green; DP. dull brown; GB. greenish brown; GY. greenish yellow; LB. light brown; OB. olive brown; RM. ripe mango colour.

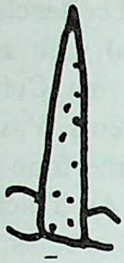

Fig. 6

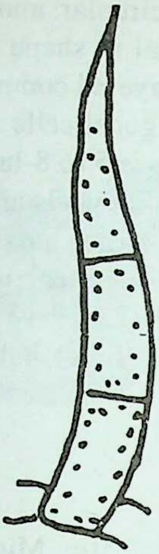

Fig. 7

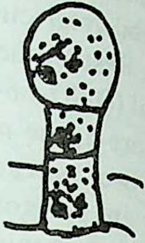

Fig. 8

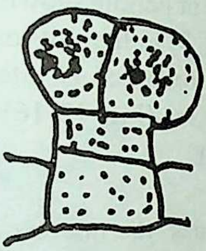

Fig. 9

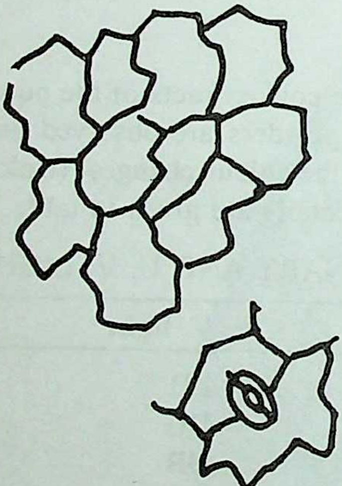

Fig. 10

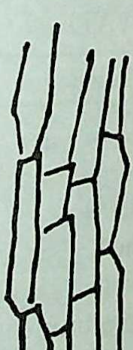

Fig. 11

Fig. 6-11

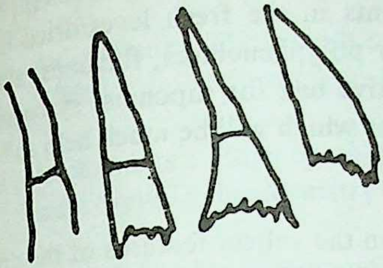

Fig. 12

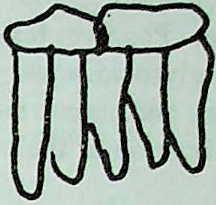

Fig. 13

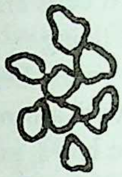

Fig. 14

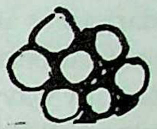

Fig. 15

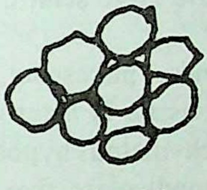

Fig. 16

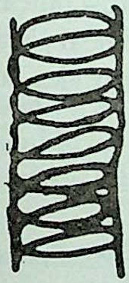

Fig. 17

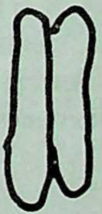

Fig. 18

Fig. 12-17

Following the simle phytochemical tests recommended by Gibbs [3] for identification of the chemical constitutents in the fresh leves/dried leaf powder. Positive tests were recorded for polyphenolases, flavones, alkaloids, glycosides and juglones, and negetive test for saponins, syrengins, acubins and aurone and leucoanthocyanins which will be much help full in identification of the taxa in powder form.

In the light of the present investigation the salient features of the drug which would be useful in identification in its different states are follows:

Leaf : Leaves ovate, base sub-cordate, apex acute; margin serrate, densely hairy, green 10-13 x 7-9 cm with Craspido-brochidodromous venation.

Secondary veins are six pairs, mostly branched. Epidermal cells are mostly polygonal anisodiametric, sides abaxially sinuate, sinuses being "v" shaped and adaxially wavy to sinuate. Costal cells distinct on all veins. Leaves are amphistomatic; stomata mostly diacytic rarely associated with anisotricytic. Unicellular and uniseriate conical hairs uniseriate capitate hairs, uniseriate peltate hairs present all over except on margin. In transection leaf is bifacial, abaxially ribbed at midvein, secondary vein, tertiary vein and feebly at adaxial midvein. Palisade is one layered and spongy mesophyll is 7-8 layered. Midvein with single large semi circular shaped vascular bundle.

Petiole : Five cm long, slender, pubescent, In transection it is circular and adaxially groved. Ground tissue is heterogenous consisting of 3-layered, compactly arranged collenchymatous hypodermis and 5-8 layered parenchymatous tissue. Vascular bundles are five; the median one is large, lunar shaped and the four small laterals two on each of the arms of median bundle.

Leaf powder : Leaf powder is brown, aromatic microscopically fragments of epidermis, stomata, mesophyll, trichomes and tracheary elements can be observed. Following the simle tests [3] positive test for glycosides, polyphenolases, alkaloids flavounes, juglones are recorded.

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EXPLANATON TO FIGURES

Plate 1. *Leonotis nepetaefolia* L. Fig. 1 & 2. Adaxial and abaxial epidermis of leaf lamina (x 440); 3. Transection of leaf lamina from midrib (x 52), lamina (x96) and margin (x 102); 4. Sectional view of petiole (x 75); 5. Venation pattern of leaf lamina. (CO. Collenchyma; 1b. loop forming branch; Pe. Phloem parenchyma; Ph. Phloem. pp. Papilate: Sc. Secretory cell ; Sp. Spongy parenchyma X. xylem; 1º. Primary vein; 2ºb. Secondary vein branched. 3º. Tertiary vein).

Figs. 6. Unicellular conical hair 7. Uniseriate conical hair; 8. Uniseriate capitate hair. 9. Uniseriate peltate hair; 10. Pieces of the epidermal cells; 11. costal cells; 12. Full or bits of trichomes; 13 & 18. Palisade parenchyma; 14. Spongy parenchyma; 15. Collenchyma cells; 16. Ground parenchyma; 17. Tracheary elements.

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THERMODYNAMICS AND STOCHASTICS OF BIOLOGICAL GROWTH

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ABSTRACT

A non-equilibrium thermodynamic model of biolgical growth together with the stochastic analysis of nonequilibrium fluctuation has been presented.

AMS (MOS) Subject classification (1980) : 92A7

Keywords : Gompertz equation, Entropy production, Quasi-stationary state, Non-equilibrium fluctuation.

INTRODUCTION

The study of growth, development and aging of living organism is an important field of biological sciences. A purely thermodynamic theory of growth and aging is inadequate in view of the stochastic nature due to internal noises irrespective of the random effect of the environment. The non-equilibrium thermodynamic model combined with the stochastic theory of non-equilibrium fluctuation is the proper physico-chemical theory of biological growth and aging [1]. In some earlier papers [2-4] we have tried to develop non-equilibrium thermodynamic and stochastic models of biological growth and aging. In the present paper we shall study the thermodynamic and stochastic behaviour of a growth model which was earlier derived from stochastic thermodynamic model of the system.

BIOLOGICAL GROWTH : THERMODYNAMIC AND STOCHASTIC MODELS

The living system is an open system. For the applicability of the thermodynamics of irreversible processes to the process of biological growth and development, we must have the entropy balance equation for open system.

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$$\frac{1}{m}\frac{dS}{dt} = \frac{1}{m}\frac{d_iS}{dt} + \frac{1}{m}\frac{d_eS}{dt} \tag{1}$$

where S is the entropy of the system at any time t, d_iS is the entropy production due to irreversible processes inside the system and d_eS is the flow of entropy due to the exchange with the environment. A fundamental quantity of non-equilibrium processes is the dissipative function ψ which is related to the rate of entropy production by the relation

$$\psi = T\frac{d_is}{dt} \tag{2}$$

where T is the temperature of the system. The basic problem of the non-equilibrium thermodynamics of biological growth and development is to find the appropriate dissipative function for the living system. In the earlier paper [2] we have derived the expression of the dissipative function ψ as

$$\psi = A\ e^{-\alpha t} \tag{3}$$

where A is a constant and α is a parameter. Following Zotin and Zotina [1], let us take the specific current weight as the thermodynamic flux

$$\frac{1}{m}\frac{dm}{dt} = J \tag{4}$$

where m is the growth parameter (weight or mass) of the system. If X be the thermodynamic force corresponding to the flux J, then we have [5]

$$\psi = JX = A\ e^{-\alpha t} \tag{5}$$

Using the linear phenomenological relation $J = LX$, we get finally

$$\frac{dm}{dt} = a\,e^{-\beta t}m \tag{6}$$

where $a = (LA)^{1/2}$, and $\beta = \alpha/2$ is the senescence parameter.

The equation (6) is the famous Gompertzian equation which describes a wide variety of growth such as tumour growth, embryonic growth, growth of animals and men [6].

Let us now study the stochastic behaviour of the Gompertz growth process described by the equation (6). For this let us extend the deterministic equation (6) to the form of the stochastic differential equation.

$$\frac{dm}{dt} = a\, e^{-\beta t} m + f(t)$$

$$= A(t)m + f(t) \qquad (7)$$

where $A(t) = a\, e^{-\beta t}$ is the time - dependent drift coefficient and $f(t)$ is the random perturbation which is assumed to be a whitenoise of intensity ε satisfying the conditions

$$< f(t) > = 0 \text{ and } < f(t_1)\, f(t_2) > = 2\, \varepsilon\, \delta\, (t_2 - t_1) \qquad (8)$$

where $< >$ represents the average over the stochastic process. The equation (7) is generalization of the famous Langevin equation with time - dependent drift coefficient.

The stochastic behaviour of the process described by equation like (7) is manifested in the moment equations or moments, particularly the second - order moment of the growth parameter $m(t)$. The non-stationary stochastic behaviour of the stochastic Gompertzian equation (7) has been studied elsewhere [4]. In this paper we are, however, interested in the quasi-stationary state of the system. Such a state is characterized by the slow variation of the drift coefficient $A(t)$ and the diffusion coefficient ε which is assumed here to be a constant. The stationary (or steady) non-equilibrium state of living systems occur at different times for differtent living organisms. As far example, for human it generally appear about the age 25 to 30. During the steady non-equilibrium state or homeostasis the growth parameter remains practically constant, but at the quasi-stationary and non-stationary states it is subject to fluctuation due to innumerous internal random effects irrespective of the external or environmental influences. The study of the stochastic behaviour or fluctuations of the growth parameter for quasi-stationary state which is very near to the stationary non-equilibrium is of great biological importance [7].

The mean square fluctuation of the growth parameter $m(t)$ satisfying the stochastic differential equation (7) is given by a generalized form of the famous Nyquist's relation [2].

$$<m^2(t)>=\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\left|\frac{A(t)}{(A(t)+i\omega)}+\frac{A(t)}{(A(t)+i\omega)^3}\right|^2 d\omega \qquad (9)$$

Using $A(t) = a\, e^{-\beta t}$, $A(t) = -\beta A(t)$ and evaluating the integrals appearing in (9), we have

$$<m^2(t)>=\varepsilon\left[\frac{1}{A(t)}-\frac{\beta}{2A^2(t)}+\frac{3\beta^2}{8A^3(t)}\right] \qquad (10)$$

Let us now use the condition of quasi-stationarity. For the problem under consideration this is satisfied as stated earlier by the criteria of slow - variation of the drift - coefficient $A(t) = a\, e^{-\beta t}$ i.e. by the the condition

$$\left|\frac{A(t)}{A(t)}\right| << 1 \, or \left|\beta\right| << 1 \qquad (11)$$

Neglecting terms containing β and its higher powers, we have

$$<m^2(t)>=\frac{\varepsilon}{A(t)}=\left(\frac{\varepsilon}{a}\right) exp\,(\beta t) \qquad (12)$$

showing that the fluctuation of the growth parameter (mass or weight) increases exponentially with time unless there is some restriction on the order of (β t). Apart from the fact the fluctuation becomnes infinitely large, there is another difficulty with the Gompertzian model equation (6). The equation (6) shows that the stationary or steady growth of the living system is reached after an infinitely large time. This is practically impossible for living organisms having terminated growth and finite life span. All these show that there must be some restriction on the order of the exponent (βt). In the next section we are going to resolve all these difficulties faced by Gompertzian Model.

A LINEAR GROWTH MODEL: THERMODYNAMICS AND STOCHASTICS

To resolve the difficulties faced by the Gompertz equation (6), we assume that

$$|\beta t| << \tag{13}$$

Under this condition, the equation (6) is approximated to

$$\frac{1}{m}\frac{dm}{dt} = J = a\beta\left[\frac{1}{\beta} - t\right] = L_0(T_m - t) \tag{14}$$

where $1/\beta = T_m$ is the maximum age, $L_0 = a\beta$ is the phenomenological or kinetic coefficient. If we take = T_m - t as proportional to the thermodynamic force corresponding to the flux J, the relation (14) becomes the linear phenomenological relation between the thermodynamic flux and force. The linear approximation of (6) neglecting quadratic and higher powers of (β t) is justified by two facts. The first is that the linear approximation makes the equation amicable to the theory of linear thermodynamics of irreversible processes and the secondly it removes the infinite time to reach the steady state of the growth and also the infinitely increasing of fluctuation as we shall see a bit latter. Above all the linear model (4) is able in explaining a wide variety of biological phenomena related to growth and development of living organism [1]. It is infact, the starting or basic equation in an early development of non-equilibrium thermodynamics of biological growth by Zotin and Zotina [1]. In this paper we have arrived at it from a more general setting of thermodynamic model of growth of living organism. In terms of the growth parameter m, the equation (14) can be written

$$\frac{dm}{dt} = a\beta(T_m - t)m \tag{15}$$

Integrating (15) and using the initial conditions

$$t = 0,\ T_m = 0,\ m = m_0 \text{ (say)} \tag{16}$$

we have

$$(T_m - t)^2 = \frac{2}{a\beta}\log\ (m_0 / m) \tag{17}$$

expressing the thermodynamic force in terms of the growth parameter (mass or weight) of the system.

The rate of entropy production then become

$$\frac{d_i S}{dt} = \frac{1}{T} JX = \frac{a\beta}{T}(T_m - t)^2 \tag{18}$$

Integrating, we have

$$\Delta_t S = (S_i)_f - (S_i)_0 = \frac{a\beta}{T}\int_0^t (T_m - t)^2\, dt \tag{19}$$

Using (17) and (18), the entropy production from the initial state O at time O to the final state f at time t can be expressed as

$$\Delta_t S = \frac{2}{T}\int_{m_0}^{m} (\log m_0 - \log m)\, dm$$

$$= \frac{2}{T}\left[(m - m_0) - m \log (m / m_0)\right] \tag{20}$$

which expresses the entropy production in terms of mass or weight of the system, the temperature T being assumed to be constant during the time $(0,t)$. The expression (20) of entropy production has important sigificance. Its mathematical derivation including its biological implication will be studied later on.

Let us now consider the stochastic behaviour of the linear equation (15). The stochastic extension of (15) is given as before by

$$\frac{dm}{dt} = a\beta(T_m - t)m + f(t) \tag{21}$$

Using either directly the formula (9) with $A(t) = a\beta(t_m - t)$ or the relation (12) with the restriction (13) (i.e. $|\beta t| << 1$) the second - order moment of $m(t)$ is given by

$$< m^2(t) > = \frac{\varepsilon}{a} \tag{22}$$

Thus, in contrast to the case of Gompertz model, the fluctuation does not show infinitely diverging character. The above fluctuation, of course, depends on the intensity ε of the noise and the growth rate α.

CONCLUSION

The paper dealing with the thermodynamic and stochastic models of biological growth can be divided into two parts discussed in section (2) and (3) respectively. The first which deals with the thermodynamic derivation of Gompertz growth and its stochastic behaviour is faced with some difficulties as to the infinite time of reaching steady - state and diverging fluctuation. These difficulties have been resolved in the second part (section (3)) which deals with the linear model derived as a linear approximation of Gompertz equation. This linear model was earlier proposed by Zotin and Zotina [1], here we have arrived at it from a general non-equilibrium setting and studied its stochastic behaviour and stressed its importance in relation to the Gompertzian model of growth of living organism. The whole analysis based on simple mathematical equations or models, there is nothing new numerical or graphical works to be added.

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ON NONLINEAR STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS INVOLVING ITÔ-CLIFFORD INTEGRALS

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ABSTRACT

The aim of this paper is to study a general class of nonlinear stochastic integrodifferential equations involving Itô-Clifford integrals. We study the problems of existence, uniqueness and stability of the solutions by using the method of successive approximation.

AMS Subject Classification (1991): 60 H 20

Keywords and Phrases: Stochastic Integrodifferential Equations, Itô-Clifford Integrals, Existence and Uniqueness, Stability, successive approximation.

INTRODUCTION

In a recent paper [1] Barnett, Streater and Wilde have introduced the concept of Itô-Clifford integral and established some basic properties of this integral. They have presented in [2] an interesting study relating to existence, uniqueness and stability properties of the solutions of a stochastic differential equation involving Itô-Clifford integral. This newly formulated stochastic integral and corresponding stochastic equations are yet in their developing stage and awaits for further attention by workers in this field. The aim of this paper is to study the properties of the solutions of a general stochastic integrodifferential equation

$$(1.1)\quad dx(t) = F(t, x(t), A_1 x(t), A_2 x(t))dt + G(t, x(t), B_1 x(t), B_2 x(t))\, dw(t) + dw(t)\, H(t, x(t), P_1 x(t), P_2 x(t))$$

where

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$$A_1 x(t) = \int_{t_0}^{t} f_1(t, s, x(s))\, ds, \quad A_2 x(t) = \int_{t_0}^{t} f_2(t, s, x(s))\, dw(s),$$

$$B_1 x(t) = \int_{t_0}^{t} g_1(t, s, x(s))\, ds, \quad B_2 x(t) = \int_{t_0}^{t} g_2(t, s, x(s))\, dw(s),$$

$$P_1 x(t) = \int_{t_0}^{t} h_1(t, s, x(s))\, ds, \quad P_2 x(t) = \int_{t_0}^{t} h_2(t, s, x(s))\, dw(s),$$

and $w(t)$ is Fermion martingale, with initial condition $x(t_0) = x_0$, $t \geq t_0$ and $x_0 \,\varepsilon\, L^2(\ell t_o)$.

As in [2] we consider the case of $L^2(R_+)$, $R_+ = (0, \infty)$ with complex conjugation. Let ℓ denote the w^*-algebra generated by the bounded operators $\{\Psi(z) = C(z) + A(z), z \,\varepsilon\, L^2(R_+)\}$ where $C(.)$, $A(.)$ are the Fock fermion creation and annihilation operators respectively and $A(z)$ is the adjoint of $C(z)$; $A(z) = C(z)^*$. For given $t \geq 0$ denote by ℓt the W^* - subalgebra of ℓ generated by the fields $\psi(z)$ for $z \,\varepsilon\, L^2(R_+)$ with ess supp $z \subseteq [0, t]$ and the completion of ℓ_t with respect to $\|\cdot\|_p$ by $L^p(\ell_t)$. Therefore, for $1 \leq p < \infty$, $L^p(\ell_t)$ may be considered as a subspace of $L^p(\ell)$. A map $x: R_+ \to L^2(\ell)$ is said to be adapted if $x(t) \,\varepsilon\, L^2(\ell_t)$ for each $t \,\varepsilon\, R_+$ and a map $Z: (R_+)^2 \, x \, L^2(\ell_t) \to L^2(\ell_t)$ is said to be adapted if for any $s \leq t$, $s, t \in R_+$ and $u \in L^2(\ell_t)$ we have $Z(t, s, u) \in L^2(\ell_t)$. Similarly a map $Y: R_+ \, x \, [L^2(\ell_t)]^3 \to L^2(\ell_t)$ is said to be adapted if for any $t \in R_+$, $u_i \,\varepsilon\, L^2(\ell_t)$, $i = 1, 2, 3$, we have $Y(t, u_1, u_2, u_3) \in L^2(\ell_t)$. In particular if $x_i : R_+ \to L^2(\ell)$, $i = 1,2,3$, $z : (R_+)^2 \, x L^2(\ell) \to L^2(\ell)$ and $Y : R_+ \, x[L^2(\ell)]^3 \to L^2(\ell)$ are adapted then so are the maps $t \to Z(t, s, x_1(s))$ and $t \to Y(t, x_1(t), x_2(t), x_3(t))$. Let $V_1, \ldots, V_n \,\varepsilon\, L^2_{10c}(R_+)$ so that for any $t \geq 0$, $v_i \chi_{[0,t]} \,\varepsilon\, L^2(R_+)$, $1 \leq i \leq n$. The Wick-ordered monomial $: \psi(v_1 \chi_{[0,t]}) \ldots \psi(v_n \chi_{[0,t]}):$ is denoted by $w(t)$. It is proved in [1., Theorem 2.2, p. 176] that $w(t)$ is an L^∞-martingale adapted to the family $\{\ell_t : t \,\varepsilon\, R_+\}$. For detailed notations and definitions see [1,2].

The general formulation of equation (1.1), which in turn contains as a special case the stochastic differential equation studied by Barnett, Streater and Wilde [2], is motivated by the study of general forms of Itô -type stochastic equations by Berger and Mizel [3], Kannan and Bharucha Reid [4] and Pachpatte [5]. In section 2 we formulate our results on the existence, uniqueness and stability of the solutions of equation (1.1) and its more general form. Section 3 deals with the proofs of our main results. The method of successive approximation is used to establish our results.

STATEMENT OF RESULTS

Before stating the main results in this section we list the necessary preliminaries from [1,2] and the hypotheses needed in our subsequent discussion.

Fix $v_1, \ldots, v_n \,\varepsilon\, L^{\infty}_{loc}(R_+)$ and suppose that $W(t) = : \psi(v_1 \chi_{[0,t]}) \ldots \psi(v_n \chi_{[0,t]}):$ for $t \,\varepsilon\, R_+$. Then $W(s)\, W^*(s) = a(s)\mathbb{1}$, where

$$(2.1) \quad a(s) = \int_0^s \ldots \int_0^s |v(s_1, \ldots, s_n)|^2 \, ds_1 \ldots ds_n,$$

with $v(s_1, \ldots, s_n) = (n!)^{1/2} \mathscr{A}(v_n \otimes \ldots \otimes v_1)(s_1, \ldots, s_n)$ see [1, p. 194]. Let μ denote the measure on R_+ given by the positive increasing function $s \rightarrow a(s)$ that is $\mu([\alpha, \beta]) = a(\beta) - a(\alpha)$ for $\alpha \leq \beta$ in R_+ and let $f : R_+ \rightarrow L^2(\ell)$ be measurable and adapted with $\int_0^t \| f(s) \|_2^2 \, d\mu(s) < \infty$. It is shown in [1] that the Itô-Clifford stochastic integral satisfies the following property,

$$(2.2) \quad \left\| \int_0^t f(s)\, dW(s) \right\|_2^2 = \int_0^t \| f(s) \|_2^2 \, d\mu(s).$$

Furthermore $\int_0^t dW(s) f(s) = \left(\int_0^t f^*(s)\, dW^*(s) \right)^*$ and $W^*(s)\, W(s) = a(s)\, \mathbb{1}$, therefore we also have

$$(2.3) \quad \left\| \int_0^t dW(s)\, f(s) \right\|_2^2 = \int_0^t \| f(s) \|_2^2 \, d\mu(s),$$

see [1,2].

The equation (1.1) is equivalent to the stochastic integral equation

$$(2.4) \quad x(t) = x_0 + \int_0^t F(s, x(s), A_1 x(s), A_2 x(s))\, ds$$

$$+ \int_0^t G(s, x(s), B_1 x(s), B_2 x(s))\, dW(s) + \int_0^t dW(s)\, H(s, x(s), P_1 x(s), P_2 x(s))$$

The integrals $A_1x(s)$, $B_1x(s)$, $P_1x(s)$ are $L^2(\ell)$ Bochner integrals and the integrals $A_2x(s)$, $B_2x(s)$, $P_2x(s)$ are Itô-Clifford integrals, see [2]. Further, the first integral on right in (2.4) is an $L^2(\ell)$ Bochner integral while the second and third integrals are Itô-Clifford integrals. In order these integrals to be well defined, it is sufficient for the maps $s \to f_1(s, \tau, x(\tau))$, $s \to g_1(s, \tau, x(\tau))$, $s \to h_1(s, \tau, x(\tau))$ and $s \to F(s, x(s), A_1x(s), A_2x(s))$, $\tau \leq s$ to be L^2 - continuous and the maps $s \to f_2(s, \tau, x(\tau))$, $s \to g_2(s, \tau, x(\tau))$, $s \to h_2(s, \tau, x(\tau))$, $s \to G(s, x(s), B_1x(s), B_2x(s))$ and $s \to H(s, x(s), P_1x(s), P_2x(s))$, $\tau \leq s$ are to be adapted and L^2 - continuous. This is the situation when $s \to x(s)$ and f_2, g_2, h_2, G and H are continuous and adapted and f_1, g_1, h_1 and F are contiuous. In order to ensure that the map $s \to x(s)$ is adapted, we assume that f_1, g_1, h_1 and F are adapted. We make use of the above assumption throughout the paper without further mention.

For convenience we list the following assumption :

(H_1) For $x, y, x_k, y_k, z_k \in L^2(\ell)$, $k = 1, 2$ and $t \in [0, T]$ there exist constants K_i, M_i, N_i, $i=1,2$ and K, M, N such that

$$\| f_i(t, s, x) - f_i(t, s, y) \|_2 \leq K_i \| x-y \|_2,$$

$$\| g_i(t, s, x) - g_i(t, s, y) \|_2 \leq M_i \| x-y \|_2,$$

$$\| h_i(t, s, x) - h_i(t, s, y) \|_2 \leq N_i \| x-y \|_2,$$

for all $0 \leq s \leq t \leq T$ and

$$\|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)\|_2 \leq K(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$
$$\|G(t, x_1, y_1, z_1) - G(t, x_2, y_2, z_2)\|_2 \leq M(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$
$$\|H(t, x_1, y_1, z_1) - H(t, x_2, y_2, z_2)\|_2 \leq N(\| x_1 - x_2 \|_2 + \| y_1 - y_2 \|_2 + \| z_1 - z_2 \|_2)$$

for all $0 \leq t \leq T$.

We now state our main result on the existence and uniqueness of the solution of equation (1.1)

Theorem 1.: Let the hypothesis (H_1) be fulfilled. Then for any $x_0 \varepsilon L^2 (\ell t_0)$ there exists exactly one L^2-continuous adapted solution of equation (1.1) on $I = [t_0, \infty)$ with $x(t_0) = x_\sigma$

The result established in Theorem 1 can be extended to the following general form of stochastic integrodifferential equation involving Itô-Clifford integrals driven by $W^{(j)}(t)$, $1 \leq j \leq r$, of the form

$$(2.5)\ dx(t) = \sum_{j=1}^{r} F^{(j)}(t, x(t), A_1^{(j)} x(t), A_2^{(j)} x(t))\, dt$$

$$+ \sum_{j=1}^{r} G^{(j)}(t, x(t), B_1^{(j)} x(t), B_2^{(j)} x(t))\, dw^{(j)}(t)$$

$$+ \sum_{j=1}^{r} dw^{(j)}(t) H(t, x(t), P_1^{(j)} x(t), P_2^{(j)} x(t))$$

for $t \in T$ with initial condition $x(t_0) = x_0 \ \varepsilon\ L^2(\ell_{t0})$
where

$$A_1^{(j)} x(t) = \int_{t_0}^{t} f_1^{(j)}(t, s, x(s))\, ds,\ A_2^{(j)} x(t) = \int_{t_0}^{t} f_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$B_1^{(j)} x(t) = \int_{t_0}^{t} g_1^{(j)}(t, s, x(s))\, ds,\ B_2^{(j)} x(t) = \int_{t_0}^{t} g_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

$$P_1^{(j)} x(t) = \int_{t_0}^{t} h_1^{(j)}(t, s, x(s))\, ds,\ P_2^{(j)} x(t) = \int_{t_0}^{t} h_2^{(j)}(t, s, x(s))\, dw^{(j)}(s),$$

and $W^{(j)}(t)$ denotes the Wick-ordered monomial

$:\psi(v_1^{(j)} \chi^{(j)}_{[0,t]}) \ldots \psi(v_n^{(j)} \chi^{(j)}_{[0,t]}):$ for $t \in R_+$. We note that $(W^{(j)}(s))\,(W^{(j)}(s))^* = a^{(j)}(s)\mathbb{1}$ where $a^{(j)}(s)$ is defined as in (2.1) with corresponding changes. Let $\mu^{(j)}$, $1 \leq j \leq r$ denote the measure on R_+ given by the positive increasing function

$s \to a^{(j)}(s)$. We assume that the functions $f_i^{(j)}, g_i^{(j)}, h_i^{(j)}, i = 1,2, j=1,2, \ldots, r$ defined on $(R_+)^2 x L^2(\ell)$ into $L^2(\ell)$ and functions $F^{(j)}, G^{(j)}, H^{(j)}$ defined on $R_+ x[L^2(\ell)]^3$ into $L^2(\ell)$ for $j = 1,2 \ldots r$ and L^2-continuous and adapted.

We make the following assumptions on the functions involved in (2.5). (H_2) for $x, y, x_k, y_k, z_k \in L^2(\ell), k = 1,2$ and $t \in [0,T]$ there exist constants $K_i^{(j)}$, $M_i^{(j)}$, $N_i^{(j)}$ and $K^{(j)}, M^{(j)}, N^{(j)}, i=1, 2, j=1,2\ldots,r$ such that

$$\| f_i^{(j)}(t, s, x) - f_i^{(j)}(t, s, y) \|_2 \le K_i^{(j)} \| x-y \|_2,$$

$$\| g_i^{(j)}(t, s, x) - g_i^{(j)}(t, s, y) \|_2 \le M_i^{(j)} \| x-y \|_2,$$

$$\| h_i^{(j)}(t, s, x) - h_i^{(j)}(t, s, y) \|_2 \le N_i^{(j)} \| x-y \|_2,$$

for all $0 \le s \le t \le T$ and

$$\|F^{(j)}(t, x_1, y_1, z_1) - F^{(j)}(t, x_2, y_2, z_2)\|_2 \le K^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|G^{(j)}(t, x_1, y_1, z_1) - G^{(j)}(t, x_2, y_2, z_2)\|_2 \le M^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

$$\|H^{(j)}(t, x_1, y_1, z_1) - H^{(j)}(t, x_2, y_2, z_2)\|_2 \le N^{(j)}(\|x_1 - x_2\|_2 + \|y_1 - y_2\|_2 + \|z_1 - z_2\|_2)$$

for all $0 \le t \le T$.

Our next theorem gives sufficient conditions for existence and uniqueness of the solution of equation (2.5).

Theorem 2. : Suppose the hypothesis (H_2) holds. Then for any $x_0 \,\varepsilon\, L^2(\ell_{t_0})$ there exists a unique L^2- continuous adapted solution of the equation (2.5) on I with $x(t_0) = x_0$.

Remark : We note that Theorems 2.1 and 2.2 established in [2] are special cases of our Theorems 1 and 2 respectively. An interesting study relating to more general Itô type stochastic integrodifferential equations can be found in recent papers by Berger and Mizel [3] and Pachpatte [5].

The following theorems deal with the stability of the solutions of equations (1.1) and (2.5).

Theorem 3.: Assume that the hypothesis (H_1) holds. Let $y(t)$ and $z(t)$ for t

$\geq t_0$ be the solutions of equation (1.1) with initial conditions $y(t_0) = x_0$ and $z(t_0) = x_0^*$ respectively with $x_0, x_0^* \in L^2(\ell_{t0})$. For given $T > t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_0 - x^*_0\|_2 < \delta$ then $\| y(t) - z(t) \|_2 < \varepsilon$ for all $t \in [t_0, T]$.

Theorem 4.: *Let the hypothesis* (H_2) *hold. Suppose* $y(t)$ *and* $z(t)$ *for* $t \geq t_0$ *are the solutions of the stochastic integrodifferential equation (2.5) with initial conditions* $y(t_0) = x_0$ *and* $z(t_0) = x_0^*$ *respectively and* $x_0, x_0^* \in L^2(\ell_{t0})$. *For given* $T > t_0$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $\| x_0 - x_0^* \|_2 < \delta$ *then* $\| y(t) - z(t) \|_2 < \epsilon$ *for all* $t \in [t_0, T]$.

Proofs of Theorems 1 and 2

Let T be fixed in I such that $t_0 \leq t \leq T$. We define

$$(3.1) \quad x^{(0)}(t) = x_0,$$

and for $n \geq 0$,

$$(3.2)\ x^{(n+1)}(t) = x_0 + \int_{t_0}^{t} F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))\, ds$$

$$+ \int_{t_0}^{t} G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))\, dw(s)$$

$$+ \int_{t_0}^{t} dw(s) H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)).$$

We first show that the process $x^{(n)}(t)$, $n \geq 1$ is well defined and it is adapted L^2- continuous on $[t_0 T]$. The functions $f_i(s, \tau, x_0)$, $g_i(s, \tau, x_0)$, $h_i(s, \tau, x_0)$, $i = 1, 2, F(s, x_0 A_1 x^{(0)}(s), A_2 x^{(0)}(s))$, $G(s, x_0 B_1 x^{(0)}(s), B_2 x^{(0)}(s))$ and $H(s, x_0, P_1 x^{(0)}(s),. P_2 x^{(0)}(s))$ are adapted and L^2-continuous for $t_0 \leq \tau \leq s \leq T$. Thus it is easy to see that $x^{(1)}(t)$ is well defined for $t_0 \leq \tau \leq s \leq T$. Also we observe from the fact that continuity of a function implies boundedness of the function on compact intervals and using the isometry property we see that $t \rightarrow x^{(1)}$ is continuous map:

$$[t_0 T] \rightarrow L^2(\ell).$$

Assume that $x^{(n)}(t)$ is adapted and L^2-continuous on $[t_0, T]$. Then, each of the functions $f_i(s, \tau, x^{(n)}(\tau))$, $g_i(s, \tau, x^{(n)}(\tau))$, $h_i(s, \tau, x^{(n)}(\tau))$, $i = 1, 2$, $F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s))$, $G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$ and $H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$ is adapted and L^2-continuous on $[t_0, T]$ and hence bounded. It can be easily seen that $x^{(n+1)}(t)$ is adapted and by using isometrics (2.2) and (2.3) we observe that $t \to x^{(n+1)}(t)$ is L^2-continuous on $[t_0, T]$. This proves that the process $x^{(n)}(t)$, $n \geq 1$ is well defined, adapted and L^2-continuous on $[t_o, T]$

We next consider the convergence of the sequence $\{x_n(t)\}$, $n \geq 0$ defined by (3.1)-(3.2). From (3.2) and the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we observe that

$$(3.3) \quad \begin{aligned} \| x^{(n+1)}(t) - x^{(n)}(t) \|_2^2 \leq\ & 3 \, \Big\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\ & - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))] \, ds \Big\|_2^2 \\ & + 3 \, \Big\| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s)) \\ & - G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \, dw(s) \Big\|_2^2 \\ & + 3 \, \Big\| \int_{t_0}^{t} dw(s) [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s)) \\ & - H(s, x^{(n-1)}(s), P_1 x^{(n-1)}(s), P_2 x^{(n-1)}(s))] \Big\|_2^2 . \end{aligned}$$

From the isometries (2.2) and (2.3) the boundedness of the wave functions v_1, ..., v_n over $[t_0, T]$, the hyposheris (H_1), $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and Schwarz inequality we observe that

$$(3.4) \quad \begin{aligned} & \Big\| \int_{t_0}^{t} [F(s, x^{(n)}(s), A_1 x^{(n)}(s), A_2 x^{(n)}(s)) \\ & - F(s, x^{(n-1)}(s), A_1 x^{(n-1)}(s), A_2 x^{(n-1)}(s))] \, ds \Big\|_2^2 \end{aligned}$$

$$\le K^2 (t - t_0) \int_{t_0}^{t} [\| x^{(n)}(s) - x^{(n-1)}(s) \|_2$$

$$+ K_1 \int_{t_0}^{s} \| x^{(n)}(\tau) - x^{(n-1)}(\tau) \|_2 \, d\tau$$

$$+ (K_2^2 Q_1^2 \int_{t_0}^{s} \| x^{(n)}(\tau) - x^{(n-1)}(\tau) \|_2^2 \, d\tau)^{1/2}]^2 \, ds$$

$$\le 3K^2 (T - t_0) Q_1^* \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s) \|_2^2 \, ds,$$

(3.5)
$$\| \int_{t_0}^{t} [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \, dw(s) \|_2^2$$

$$= \int_{t_0}^{t} \| [G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s))] \|_2^2 \, d\mu(s)$$

$$\le C_1^2 \int_{t_0}^{t} \| G(s, x^{(n)}(s), B_1 x^{(n)}(s), B_2 x^{(n)}(s))$$

$$- G(s, x^{(n-1)}(s), B_1 x^{(n-1)}(s), B_2 x^{(n-1)}(s)) \|_2^2 \, ds$$

$$\le 3 C_1^2 M^2 Q_2^* \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s) \|_2^2 \, ds,$$

and

(3.6)
$$\| \int_{t_0}^{t} dW(s) \, [H(s, x^{(n)}(s), P_1 x^{(n)}(s), P_2 x^{(n)}(s))$$

$$- H\,(s,\, x^{(n-1)}(s),\, P_1 x^{(n-1)}(s),\, P_2\, x^{(n-1)}(s))]\|\|_2^2$$

$$= \int_{t_0}^{t} \|\, H\,(s,\, x^{(n)}(s),\, P_1 x^{(n)}\,(s),\, P_2\, x^{(n)}(s))$$

$$- H\,(s,\, x^{(n-1)}(s),\, P_1 x^{(n-1)}(s),\, P_2\, x^{(n-1)}(s))\,\|_2^2\, d\mu(s)$$

$$\leq C_2^{\,2} \int_{t_0}^{t} \|H\,(s,\, x^{(n)}(s),\, P_1 x^{(n)}\,(s),\, P_2\, x^{(n)}(s))$$

$$- H\,(s,\, x^{(n-1)}(s),\, P_1 x^{(n-1)}(s),\, P_2\, x^{(n-1)}(s))\,\|_2^2\, ds$$

$$\leq 3C_2^{\,2} N^2 Q_3{}^* \int_{t_0}^{t} \|x^{(n)}(s) - x^{(n-1)}\,(s)\,\|_2^2\, ds,$$

where

$$Q_1^* = \left[1 + \frac{K_1^2\,(T - t_0)^2}{2} + Q_1^2\, K_2^2\,(T - t_0) \right]$$

$$Q_2^* = \left[1 + \frac{M_1^2\,(T - t_0)^2}{2} + Q_2^2\, M_2^2\,(T - t_0) \right]$$

$$Q_3^* = \left[1 + \frac{N_1^2\,(T - t_0)^2}{2} + Q_3^2\, N_2^2\,(T - t_0) \right]$$

and C_1, C_2, Q_1, Q_2 and Q_3 are constants depending on t_0 and T. Using (3.4) - (3.6) in (3.3) we get

(3.7) $$\| x^{(n+1)}(t) - x^{(n)}(t)\, \|_2^2 \leq D \int_{t_0}^{t} \| x^{(n)}(s) - x^{(n-1)}(s)\, \|_2^2\, ds,$$

where

(3.8) $$D = 9\, [\, K^2\, (T - t_0)\, Q_1{}^* + C_1{}^2\, M^2\, Q_2{}^* + C_2{}^2\, N^2\, Q_3{}^*].$$

By succesive iterations on (3.7) we have

$$\| x^{(n+1)} (t) - x^{(n)} (t)\|_2^2$$

$$\leq D^n \int_{t_0}^{t} \int_{t_0}^{s_{n-1}} \cdots \int_{t_0}^{s_1} \| x^{(1)} (s_{n-1}) - x^{(o)} (s_{n-1})\|_2^2 \, ds \, ds_1 \ldots ds_{n-1}$$

We note that both $s \to x^{(1)} (s)$ and $s \to x^{(0)} (s) = x_0$ *are* L^2 - continuous and hence bounded on $[t_0, T]$. Thus by using Cauchy integral formula, we get

$$\| x^{(n+1)} (t) - x^{(n)} (t)\|_2^2 \leq D_1^n \frac{(t - t_0)^n}{n!}, \tag{3.9}$$

where D_1 is a positive constant depending on t_0 and T. From the inequality (3.9), for any $n > r$, it follows that

$$\| x^{(n+1)} (t) - x^{(r+1)} (t)\|_2 = \| \sum_{m=r+1}^{n} (x^{(m+1)}(t) - x^{(m)} (t)\|_2$$

$$\leq \sum_{m=r+1}^{n} \left[D_1^m \frac{(t - t_0)^m}{m!} \right]^{1/2}$$

Thus, we observe that there is an $x(t)$ in $L^2(\ell)$.such that $x^{(n)} (t)$ converges uniformly to $x(t)$ in $L^2(\ell)$ on $[t_0, T]$. Since each $x^{(n)} (t)$ is L^2 - continuous and adapted, the process $x(t)$ is L^2 - continuous and adapted on $[t_0, T]$.

Finally we show that the process $x(t)$, $t \geq t_0$ satisfies equation (1.1).. It is clear that $x(t_0) = x_0$. Also

$$\| \int_{t_0}^{t} [F (s, x^{(n)}(s), A_1 x^{(n)} (s), A_2 x^{(n)}(s)) \tag{3.10}$$

$$- F (s, x(s), A_1 x(s), A_2 x(s))] \, ds \|_2^2$$

$$\leq 3K^2 (T - t_0)\, Q_1{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty$$

since, for $s \varepsilon [t_0, T]$, $x^{(n)}(s) \to x(s)$ uniformly in $L^2(\ell)$. Further we note that

$$(3.11) \qquad \| \int_{t_0}^{t} [G (s, x^{(n)}(s), B_1 x^{(n)} (s), B_2 x^{(n)}(s))$$

$$- G (s, x (s), B_1 x (s), B_2 x (s))]\, dW(s)\|_2^2$$

$$\leq 3C_1{}^2 M^2 Q_2{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty.$$

and

$$(3.12) \qquad \| \int_{t_0}^{t} dW (s)[H(s, x^{(n)}(s), P_1 x^{(n)} (s), P_2\, x^{(n)}(s))$$

$$- H (s, x(s), P_1 x(s), P_2\, x(s))] \|_2^2$$

$$\leq 3C_2{}^2 N^2 Q_3{}^* \int_{t_0}^{t} \| x^{(n)}(s) - x(s)\|_2^2 ds$$

$$\to 0 \text{ as } n \to \infty .$$

Thus from (3.2) and (3.10), - (3.12) it follows that

$$(3.13) \qquad x(t) = \lim_{n\to\infty} x^{(n+1)}(t)$$

$$= \lim_{n\to\infty} [x_0 + \int_{t_0}^{t} F (s, x^{(n)}(s), A_1 x^{(n)} (s), A_2\, x^{(n)}(s))ds$$

$$+ \int_{t_0}^{t} G\,(s,\, x^{(n)}(s),\, B_1 x^{(n)}(s),\, B_2\, x^{(n)}(s))\; dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\; H(s,\, x^{(n)}(s),\, P_1 x^{(n)}(s),\, P_2\, x^{(n)}(s))\;]$$

$$= x_0 + \int_{t_0}^{t} F\,(s,\, x(s),\, A_1\, x\,(s),\, A_2\, x(s))ds$$

$$+ \int_{t_0}^{t} G\,(s,\, x(s),\, B_1 x(s),\, B_2\, x(s))\; dW(s)$$

$$+ \int_{t_0}^{t} dW(s)\; H(s,\, x(s),\, P_1\, x(s),\, P_2\, x(s))$$

This shows that $x(t)$, $t \geq t_0$ is a solution of the equation (1.1) with initial condition $x(t_0) = x_0$ for $t\ \varepsilon\ [t_0, T]$. Clearly $T > t_0$ is arbitrary, hence we can determine a solution $x(t)$ for all $t \geq t_0$.

We next prove the uniqueness of solutions of the equation (1.1). Let $x(t)$ and $y(t)$ be two L^2-continuous and adapted solutions of (1.1) such that $x(t_0) = y(t_0) = x_0$. By using $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, isometries (2.2) and (2.3), Schwartz inequality, hypothesis (H_1) and boundedness of $V_1,...V_n$ on $[t_0\,, t]$ we observe that

$$(3.14)\quad \|x(t) - y(t)\|_2^2 \leq 3(t - t_0)\int_{t_0}^{t} \|\, F(s,\, x(s),\, A_1\, x(s),\, A_2\, x(s))$$

$$- F(s,\, y(s),\, A_1\, y(s),\, A_2\, y(s))\|_2^2\; ds$$

$$+\, 3\int_{t_0}^{t} \|\, G(s,\, x(s),\, B_1\, x(s),\, B_2\, x(s))$$

$$- G(s, y(s), B_1 y(s), B_2 y(s))\|_2^2 \, d\mu(s)$$

$$+ 3\int_{t_0}^{t} \| H(s, x(s), P_1 x(s), P_2 x(s))$$

$$- H(s, y(s), P_1 y(s), P_2 y(s))\|_2^2 \, d\mu(s)$$

$$\leq D \int_{t_0}^{t} \| x(s) - y(s)\|_2^2 \, ds,$$

Where D is as defined in (3.8). Thus from Gronwall's inequality it follows that

$$\|x(t) - y(t)\|_2^2 = 0, \quad t \, \varepsilon \, [t_0, T]. \tag{3.15}$$

Since both $x(t)$ and $y(t)$ are L^2- continuous, from (3.15) we get

$$\|x(t) - y(t)\|_2 = 0, \tag{3.16}$$

for all $t \in [t_0, T]$. Thus $x(t) = y(t)$, for each $t \in [t_0, T]$. This completes the proof of Theorem 1. The proof of Theorem 2 follows by arguments as in the proof of Theorem 1 with suitable modifications. We omit the details.

Proofs of Theorems 3 and 4

We define $y^{(n+1)}(t)$ and $z^{(n+1)}(t)$ for $n = 0, 1 \ldots$ with $y^{(0)}(t) = x_0$ and $z^{(0)}(t) = x_0^*$ similar to $x^{(n+1)}(t)$ with $x^{(0)}(t) = x_0$ defined in the proof of Theorem 1 for all $t \, \varepsilon I$. By using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, the isometries (2.2) and (2.3), the Schwarz inequality, the boundedness of the wave functions $\nu_1, \ldots, \nu_n$ over $[t_0, T]$, the hypothesis (H_1) we observe that

$$\| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \leq 4 \, \|x_0 - x_0^*\|_2^2 \tag{4.1}$$

$$+ 4D^* \int_{t_0}^{t} \| y^{(n)}(s) - z^{(n)}(s)\|_2^2 \, ds,$$

where $D^* = D/3$ and D is as defined in (3.8). From (4.1) by successive iterations, we get

$$\text{(4.2)} \quad \| y^{(n+1)}(t) - z^{(n+1)}(t)\|_2^2 \leq 4 \sum_{m=0}^{n} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0{}^*\|_2^2$$

$$+ (4D^*)^{n+1} \int_{t_0}^{t} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_n} \|y^{(0)}(s_{n+1}) - x^{(0)}(s_{n+1})\|_2^2 ds_{n+1} \ldots ds_1$$

$$\leq 4 \sum_{m=0}^{n+1} \frac{(4D^*)^m}{m!} (t - t_0)^m \|x_0 - x_0{}^*\|_2^2 \leq 4 \|x_\upsilon - x_0{}^*\|_2^2 e^{4D^*T},$$

for all t, $t_0 \leq t \leq T$ and the result follows. this completes the proof of Theorem 3. The proof of Theorem 4 follows by similar argument as in the proof of Theorem 3 with suitable modifications, and hence we omit the details.

Remark 2.

We note that, one can very easily establish convergence theorems for the solutions of equations (1.1) and (2.5) similar to the Theorem 5.2 given in [2] with suitable modifications. In concluding this paper we note that regarding the study of stochastic differential equations involving Itô-Clifford integrals many problems are still open. The authors have tried to obtain a formula similar to Itô formula for stochastic differential equations involving Itô-Clifford integrals but could not get satisfactory answer. Once such a formula is available it will give rise to the study of many problems relating to stochastic equations involving Itô-Clifford integrals. The study of boundary value problems and various properties of second and higher order stochastic differential equations involving Itô-Clifford integrals awaits for its development.

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REGULARLY PERIODIC ELEMENTS OF A GROUP RING

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ABSTRACT

In this note the author continues the study of regularly periodic elements of a group ring. A necessary and sufficient condition for an element to be regularly periodic in a group ring is obtained. For more properties about regularly periodic elements of a ring please refer [1].

Definition 1

Let R be an arbitrary associative ring. An element x of R is said to be a Regularly periodic element of periodicity n, $(n \geq 2)$ if there exists n distinct elements in R different from x say $r_1, r_2, \dots r_n$ $(r_1, r_2, \dots r_n \in R)$ such that

$$\begin{aligned}
x^2 &= r_1 = x^{n+2} = x^{2n+2} = \dots \\
x^3 &= r_2 = x^{n+3} = x^{2n+3} = \dots \\
x^4 &= r_3 = x^{n+4} = x^{2n+4} = \dots \\
&\cdot \\
&\cdot \\
&\cdot \\
x^n &= r_{n-1} = x^{2n} = x^{3n} = \dots \\
x^{n+1} &= r_n = x^{2n+1} = x^{3n+1} = \dots
\end{aligned}$$

Throughout this paper K denotes a field and G a group: KG the group ring of G over K.

Proposition 1:

Let G be a finite group and K any field of characteristic zero. Then the group ring KG has regularly periodic elements.

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Proof.

Given G is finite, let $|G| = n$. *Consider* $\alpha = 1/n\ \Sigma\, g_i\ (g_i \varepsilon\, G)$ clearly $\alpha \in KG$ with $\alpha^2 = \alpha$. Hence KG has regularly periodic elements.

Proposition 2 :

Let G be any group having an element of finite order and K a field of characteristic zero. Then KG has a regularly periodic element. **Proof.** Let $g \varepsilon\, G$ be such that $g^m = 1\ (m < \infty)$. Consider $\alpha = \dfrac{1}{m} \displaystyle\sum_{i-0}^{m-1} g^{1}$ in KG. Clearly α is a periodic element of KG.

Proposition 3.

Let G be a torsion free abelian group and K any field; No nontrivial element of KG is regularly periodic.

Proof.

G is torsion free abelian hence KG is a domain. Therefore KG cannot contain even an idempotent. Thus KG has no nontrivial regularly periodic element.

Proposition 4.

Let G be a finite group and K any field. If $H_1, H_2, \ldots, H_n$ are subgroups of G, then the group ring KG has periodic elements.

Proof.

For each subgroup H_i of G. $\alpha = \dfrac{1}{|H_i|}\ \Sigma h_i\ (h_i\ \varepsilon H_i)$ is a regularly periodic element of KG. (H_i) denotes the number of elements on H_i).

Proposition 5.

Let $Z_2 = (0, 1)$ and S_n be the symmetric group of degree n. Then every element of Z_2S_n which is not nilpotent is regularly periodic.

Proof.

Since order of Z_2S_n is $2^{n!}$ Z_2S_n is a finite ring and hence every non nilpotent element is regularly periodic.

Theorem 6.

Let G be an abelian group. The group ring KG has a nontrivial regularly periodic element if and only if G has an element of finite order.

Proof.

If G has an element of finite order by Proposition 2 the group ring KG has a regularly periodic element.

Conversely if KG has a regularly periodic element to prove G has an element of finite order. Let $\alpha = \Sigma\alpha_i g_i \varepsilon KG$ be regularly periodic; this implies there exists n distinct elements $\beta_1, \beta_2, \ldots\ldots\ldots\ldots \beta_n \varepsilon KG$ with

$$\alpha^2 = \beta_1 = \alpha^{n+2} = \alpha^{2n+2} = \ldots$$
$$\alpha^3 = \beta_2 = \alpha^{n+3} = \alpha^{2n+3} = \ldots$$
$$\alpha^4 = \beta_3 = \alpha^{n+4} = \alpha^{2n+4} = \ldots$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\alpha^n = \beta_{n-1} = \alpha^{2n} = \alpha^{3n} = \ldots$$
$$\alpha^{n+1} = \beta_n = \alpha^{2n+1} = \alpha^{3n+1} = \ldots$$

Clearly $\alpha^2(\alpha^n-1) = 0$ this implies $\alpha^n = 1$ or KG has divisors of zero. Now KG has no divisor of zero only when G is not torsion free. That is there are elements in G of finite order. $\alpha^n = 1$ is impossible if every element in G is torsion free thus G has elements of finite order. Hence the theorem.

Remark .

If we do not assume G is abelian we may not be able to conclude KG has no zero divisors if every element of G is torsion free. Thus the zero divisor conjecture for group rings can be reformulated as follows.

Conjecture.

Let G be a torsion free non abelian group and K any field. Can KG have regularly periodic elements?

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FIXED POINTS FOR A PAIR OF DENSIFYING MAPPINGS

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ABSTRACT

In the present paper we establish a fixed point theorem for a pair of densifying mappings which includes the results or Khan and Fisher [2] and Furi and Vignoli [1].

Keywords and Phrases : Fixed point, densifying group,

Mathematics Subject classifications (1991) : 54 H25, 47 H10

INTRODUCTION

Kuratowski [3] introduced the concept of **measure of non-compactness** of a bounded set in a metric space as follows :

Defination 1 : Let A be a bounded set in a metric space (X,d). Then the **measure of non-compactness** of A, written as $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A admits a finite covering by sets with diameter less than ε.

Definition 2 : [1] A mapping T of a metric space (X,d) into itself is said to be **densifying** if for every bounded subsets A of X with $\alpha(A) > O$.

$$\alpha(T(A)) < \alpha(A)$$

Reich [4] obtained the following theorem.

Theorem A : Let T be a mapping of a compact metric space X into itself such that

$$d(Tx, Ty) < (1/2)[d(x, Tx) + d(y,Ty)]$$

for all $x,y \in X$ with $Tx \neq Ty$, then T has a unique fixed point.

MAIN RESULT

We present the following theorem.

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Theorem : Let S and T be continuous densifying mappings of a complete metric space (X,d) into itself such that for all x,y in X

$$F_1(Sx,TSy) < max\ \{[F_2(x,Sy), F_2(x,Sx), F_2(y,Sy), (1/2)F_2(x,Sx)$$

$$+ (1/2)F_1(Sy,TSy)]\ U\ min\ [F_2(x,TSy),\ F_2(Sy,Sx)]\}$$

and

$$F_2\,(Tx,STy) < max\{[F_1(x,Ty), F_1(x,Tx), F_1(y,Ty), (1/2)F_1(x,Tx)$$

$$+ (1/2)\ F_2(Ty,STy)]\ U\ min[F_1(x,STy), F_1(Ty,Tx)]\}$$

where F_1 and F_2 are real-valued functions of $X \times X$ into $[0, \infty)$ with either F_1 or F_2 lower semi-continuous. If for some x_o in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2x+1} = x_{2n+2}$ for $n = 0,1,2,.......$ is bounded and $F_1\,(x,x) = F_2(x,x) = 0$ for all $x \in X$, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique common fixed point of S and T.

Proof : We may assume that F_1 is lower semi-continuous. Consider the set $A = \{x_{2n+1} : n = 0,1,2, \ldots\}$. Then $ST(A) = \{x_{2n+1} : n = 1,2,........\}$ and so $ST(A) \subset A$. The continuity of S and T now implies that $ST\,(\overline{A}) \subset \overline{ST\,(A)}$. Now suppose that $\alpha\ (A) > 0$. Then

$$\alpha\ (\mathrm{A}) = \alpha\ (\{x_1\}\ U\ ST(A))$$

$$= max\ \{\ \alpha\ (x_1),\ \alpha\ (ST(A))\}$$

$$= \alpha\ (ST(A)) < \alpha\ (A)$$

giving a contradiction. This contradiction implies that $\alpha\ (A) = 0$ and so A is relatively compact. Since, the function F_1 is lower semi continuous, the function f defined by

$$f(x) = F_1\ (x,Tx)$$

for all x in $\overline{A}$ is lower semi-continuous and so assumes its minimum value at some z in $\overline{A}$. Thus if $Tz \neq z$ and $STz \neq Tz$ then $STz \in \overline{A}$ and

$$f(STz) = F_1\ (STz,\ TSTz)$$

$< \max\{[F_2(Tz, STz), F_2(Tz, STz), F_2(Tz, STz), (1/2) F_2(Tz, STz)$

$+ (1/2) F_2(STz, TSTz)] \cup \min [F_2(Tz, TSTz), F_2(STz, STz)]\}$

$= \max\{F_2(Tz, STz), (1/2) F_2(Tz, STz) + (1/2) F_1(STz, TSTz)\}$

We have either

$f(STz) < F_2(Tz, STz)$ or $f(STz) < (1/2) F_2(Tz, STz) + (1/2) F_1(STz, TSTz)$

which implies

$f(STz) < F_2(Tz, STz)$

Therefore

$f(STz) < F_2(Tz, STz)$

$< \max\{[F_1(z,Tz), F_1(z,Tz), F_1(z,Tz), (1/2)F_1(z, Tz)$

$+(1/2)F_2(Tz, STz)] \cup \min [F_1(z,STz), F_1(Tz,Tz)]\}$

$= \max\{F_1(z,Tz), (1/2)F_1(z,Tz) + (1/2) F_2(Tz, STz)\}$

which implies

$f(STz) < F_1(z,Tz) = f(z)$

contradicting the definition of z. It follows that either $Tz = z$, in which case z is a fixed point of T, or $STz = Tz$, in which case $z' = Tz$ is a fixed point of S.

Now suppose that z is a common fixed fixed point of S and T. Then if S and T have a second distinct common fixed point z' then $z' \neq Sz'$, $z \neq Tz'$, and so.

$F_1(z,z') = F_1(Sz, TSz')$

$< \max\{[F_2(z,Sz'), F_2(z,Sz), F_2(z',Sz'), (1/2) F_2(z,Sz)$

$+(1/2) F_1(Sz', TSz')] \cup \min [F_2(z,TSz'), F_2(Sz',Sz)]\}$

$= F_2(z,Sz')$

$= F_2(Tz, STz')$

$< \max\{[F_1(z,Tz'), F_1(z,Tz), F_1(z', Tz'), (1/2) F_1(z,Tz)$

$+ (1/2) F_2(Tz',STz')] \cup \min [F_1(z,STz'), F_1(Tz',Tz)]\}$

$= F_1(z,z')$

giving a contradiction. The common fixed fixed point z must therefore be unique. This completes the proof of the theorem.

Corollary 1 [2] : Let S and T be continuous densifying mappings of a complete metric space (X,d) into itself such that for all x,y in X

$$F(Sx,\ TSy) < F(x,Sy),\ x \neq Sy$$

and

$$F(Tx,\ STy) < F(x,Ty),\ x \neq Ty,$$

where F is a real valued lower semi-continuous function fo XxX into $[o,\infty]$. If for some x_0 in X, the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,\ldots$ is bounded, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then it is the unique fixed point of S and T.

Corollary 2 [1] : Let T be a continuous densifying mapping of a complete metric space (X,d) into itself and let F, be a real valued lower semi-continuous function of XxX into $[0,\ \infty]$ such that for all x,y in X.

$$F(Tx,Ty) < F(x,y),\ x \neq y$$

If for some x_0 in X the sequence $\{x_n\}$ defined by $Tx_{2n} = x_{2n+1}$ for $n = 0,1,2,\ldots\ldots$ is bounded, then T has a unique fixed point z.

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AN ALTERNATIVE ESTIMATOR FOR THE MEAN OF FINITE POPULATION USING AUXILIARY VARIABLES

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ABSTRACT

The supplementary information on auxiliary variables is very useful at the estimation stage in sample surveys. Several estimators developed using the supplementary information on the auxiliary variables are available in the literature. In this paper an alternative estimator of the population mean ($\overline{Y}$) using supplementary information on two auxiliary variables x (positively correlated with y) and z (negatively correlated with y) has been suggested. In section 2 the bias and the mean squared - error (MSE) of the proposed estimator have been obtained and in section 3 efficiency of the proposed estimator has been compared with the usual unbiased estimator $\overline{y}$ and with that of Srivastava [3] under simple random sampling with replacement (SRSWR).

Mathematics Subject Classification (1991) : 62 D 05

Keywords: Auxiliary variable, mean square error, ratio estimator, simple random sampling with replacement (SRSWR), unbiased estimator

INTRODUCTION

In large sample surveys, data on more than one auxiliary variables are often collected. Some of these variables may be correlated with the main character under study (y). Several successful attempts have been made to use multi-auxiliary information in increasing the precision of estimator of the population mean $\overline{Y}$. Olkin [1] extended the ratio method of estimation to incorporate supplementary information on several auxiliary variates each positively correlated with y. The case of multi-auxiliary variates, some positively correlated with y and the others negatively correlated with it, has been discussed by Srivastava [3,4] and Rao and Mudholkar [2].

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Let a simple random sample of size n is drawn from a population of size N. Let $\overline{Y}$, $\overline{X}$ and $\overline{Z}$ be the population and $\overline{y}$, $\overline{x}$ and $\overline{z}$ be the sample means of the characters y, x (positively correlated with y) and z (negatively correlated with y) respectively. Two estimators of the population mean $\overline{Y}$ based on the auxiliary information are well known in the literature, namely, the ratio estimator $\overline{y}(\overline{X}/\overline{x})$ and the product estimator $\overline{y}\ (\overline{z}/\overline{Z})$. Srivastava [3], however, suggested an estimator for $\overline{Y}$ using the supplementary information on both the variables x and z, as

$$\overline{y}_{Rw} = \overline{y}\ [w_1(\overline{X}/\overline{x}) + w_2(\overline{z}/\overline{Z})] \tag{1}$$

where w_1 and w_2 are the weights associated with the two estimators such that $w_1 + w_2 = 1$ and studied its properties.

Further let $\overline{x}^*$ and $\overline{z}^*$ be the means corresponding to the $(N - n)$ units no included in the sample. An alternative estimator ($\overline{y}_{Rw'}$) of $\overline{Y}$ associated with x and z can be constructed which is linear in $(u/U)^{\delta u}$ where u represents x, z; U represents X, Y and $\delta_u = sign\ (\rho_{yu})$. In other words.

$$\overline{y}_{Rw'} = \overline{y}[\ w'_1(\overline{x}^*/\ \overline{X}) + w'_2(\overline{Z}\ /\ \overline{z}^*\ )] \tag{2}$$

Where w_1 and w_2 are the weights such that $w'_1 + w'_2 = 1$

BIAS AND MEAN SQUARED ERROR OF $\overline{y}_{Rw'}$

Following Sukhatme and Sukhatme [5], the bias and the MSE of the estimator $\overline{y}_{Rw'}$ to the order of $O\ (n^{-1})$ can easily be seen to be,

$$B(\overline{y}_{Rw'}) = g\overline{Y}\,n^{-1}[w'_2(g\ C_z^{\ 2} + C_{zy}\ ) - w'_1C_{xy}\,] \tag{3}$$

and

$$\begin{aligned} MSE\ (\overline{y}_{Rw'}) = \ & \overline{Y}^{\,2}n^{-1}[w_1{}'^2(g^2C_x^{\ 2}+ C_y^{\ 2}- 4gC_{xy}) + w'_2{}^2(3g^2C_z^{\ 2}+ C_y^{\ 2}+ 4gC_{zy}) \\ & + 2w_1'w_2'(g^2\,C_x^{\ 2} - g^2\,C_{xz}^{\ \ 2} + C_y^{\ 2} + 2gC_{zy}- 2gC_{xy}) \\ & -2w'_2(g^2C^2_{\ z}+gC_{zy}) + 2w_1'\,g\,C_{xy}\,] \end{aligned} \tag{4}$$

where,

$$g = n/(N\text{-}n),\ C_x^{\ 2} = \sigma_x^{\ 2}\ /\ \overline{X}^{\,2},\ C_y^{\ 2} = \sigma_y^{\ 2}\ /\ \overline{Y}^{\,2},\ C_z^{\ 2} = \sigma_z^{\ 2}\ /\ \overline{Z}^{\,2},$$

$$C_{xy} = \rho_{xy}\,C_xC_y,\ C_{zy} = \rho_{xy}C_zC_y,\ \ C_{xz} = \rho_{xz}C_xC_z\,,$$

$$\rho_{xy} = Cov(x,y)/\sigma_x\sigma_y, \quad \rho_{yz} = Cov(y,z)/\sigma_y\sigma_z, \quad \rho_{xz} = Cov(x,z)/\sigma_x\sigma_z$$

The optimum weights obtained by minimizing (4) subject to the condition that $w'_1 + w'_2 = 1$ can be seen to be

$$w'_1 = \frac{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \quad \text{and}$$

$$w'_2 = \frac{g(C_x^2 + C_{xz}) + C_{xy} + C_{yz}}{g(C_x^2 + C_z^2 + 2C_{xz})} \tag{5}$$

The optimum values of bias and MSE can be obtained by substituting the values of w'_1 and w'_2 in equations (3) and (4) as,

$$B_0(\bar{y}_{Rw'}) = \frac{g\bar{Y}}{n}\left[\left(gC_z^2 + C_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}(gC_z^2 + C_{zy} + C_{xy})}{g(C_x^2 + C_z^2 + 2C_{zx})}\right] \tag{6}$$

and

$$M_0(\bar{y}_{Rw'}) = \frac{\bar{Y}^2}{n}\left[\left(g^2C_z^2 + C_y^2 + 2gC_{zy}\right) - \frac{\{g(C_z^2 + C_{xz}) + C_{xy} + C_{yz}\}^2}{g(C_x^2 + C_z^2 + 2C_{xz})}\right] \tag{7}$$

In practice, the population coefficients of variation (C_y, C_x, C_z) and the correlation coefficients (ρ_{xy}, ρ_{zy} and ρ_{xz}) are not known so that optimum weights will have to be estimated from the sample values. However, what may be known in practice may be the nature of the correlation between the variables y, x and z. This knowledge is essential for constructing the estimator. The sample estimates of optimum weights can be obtained by replacing each term in (5) by their sample estimates.

For the sake of simplicity, we may assume that

$$C_x = C_z \text{ and } \rho_{yx} = -\rho_{yz} \tag{8}$$

so that

$$w'_1 = w'_2 = ½ \tag{9}$$

From (6), (7), (8) and (9) the optimum bias and MSE of $\bar{y}_{Rw'}$ to the order of $O(n^{-1})$ can be found to be,

$$B^*_0(\bar{y}_{Rw'}) = g\bar{\bar{Y}}(2n)^{-1} C_x(gC_x - 2\rho_{xy}C_y) \tag{10}$$

and

$$M_0^*(\bar{y}_{Rw'}) = \bar{Y}^2 n^{-1}[C_y^2 + (g^2C_x^2/2)(1-\rho_{xz}) - 2gC_xC_y\rho_{xy}] \tag{11}$$

The estimator $\bar{Y}_{Rw'}$ is unbiased when $g = 2\rho_{yx}C_y/C_x$

It may be noted that when $w_1 = w_2 = 1/2$, Srivastava [3] obtained the minimum MSE of the estimator $\bar{y}_{Rw'}$ as,

$$M_0(\bar{y}_{Rw}) = \bar{Y}^2/n^{-1}[C_y^2 + (C_x^2/2)(1-\rho_{xz}) - 2\rho_{xy}C_xC_y] \quad . \tag{12}$$

EFFICIENCY COMPARISONS

The variance of the usual unbiased estimator $\bar{y}$ is given by

$$V(\bar{y}) = \bar{Y}^2 C_y^2 / n$$

From (11), (12), and (13) it can easily be shown that $\bar{y}_{Rw'}$ will be more efficient than $\bar{y}$ and $\bar{y}_{Rw}$ when

$$g/4 \le \rho_{yx} C_y / \{(1-\rho_{xz}) C_x\} \le (1+g)/4$$

Which is true when $(1-g) \ge 0$, that is when $n \le N/2$. This condition is quite obvious as the proposed estimator is dual of the estimator proposed by Srivastava [3].

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RELIGIOSITY AND LOCUS OF CONTROL AMONG COLLEGE STUDENTS

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ABSTRACT

Present investigation attemptes to study religiosity and locus of control among college students. The sample comprises of 200 college students (100 males and 100 females)po selected from Hardwar District by incidental sampling. Results reveal that female are more religious than male students, and sex differences exist in regard to locus of control among college students.

Mathematical Subject classifications (1991) : 92 J 99

Keywords & phrases : Religiosity,

INTRODUCTION

Religion is a powerful institution which plays an important role in shaping the social behaviour. In the nineteenth and twentieth centuries numerous ideas have been put forward regarding psychological and sociological factors that are responsible for the origin of religion.

The most important of these are (i) the Marxian theory which maintains that religion is one of the ideological reflection of the current state of economic interactions in a society, (ii) the similar, but more elaborately developed theory of the sociologist Emile Durkheim holds that religious beliefs constitute a projection of the structure of society, and (iii) the Freudian theory which emphasises that religious beliefs arise from projections designed to alleviate certain kinds of unconscious conflicts (7).

Galloway (8) has defined religion as the faith in a power beyond himself whereby he seeks to satisfy emotional needs and gain stability of life, which he expresses in the act of worship and service. The real religion is equally acceptable to all minds, and is almost equally philosophic, emotional, mistic and conducive to action. Williams and Cole (31) found in their study that

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religious persons carry less feeling of insecurity than non-religious persons. Mayo, et al. [16] concluded on the basis of their study that the religious men are significantly less depressed, less schizophrenic and less psychopathic deviant than non-religious ones. Adjustment and social responsibility were found to be possible correlates of religiousness (cf. [32]). Tiwari, Singh and Srivastava [28] found that high religious value oriented persons have better pattern of adjustment than low religious value oriented persons. In a study, Bhushan [3] found that religious girl students were significantly better adjusted than non-religious girl students.

Hassan and Khalique [10] showed that muslims tended to have high degree of religiousity in comparision to Hindus. Status (high/low), and sex (male/female) did not influence the religious attitudes of either Hindus or Muslims. Religiosity was found to have significant positive correlations with anxiety, rigidity and intolerance of ambiguity. Verma and Upadhyay [30] found that high religious students showed less social distance with others, if ethnic groups were treated as a whole. However, the students in general did show quite different attitudes towards various ethnic groups, and this is equally true with low and high religiosity groups, when treated separately. Sinha and Jha [27] concluded that correlation between religiosity and the total as well as the area-wise scores of religious prejudices revealed no significant relationship between the two, and high and low religous persons did not differ significantly on their religious prejudice scores.

The concept of locus of control was developed by Phares [19] in relation to beliefs regarding internal versus external control of reinforcement. It is assumed that individuals develop a general expectancy regarding their ability to control their lives. People who believe that the events that occur in their lives are a result of their own behaviour and/or personality characteristics are said to have an "expectancy of internal control", while people who believe events in their lives to be a function of luck, chance, fate, powerful others or powerful beyond their control or comprehension are said to have an "expectancy of external control".

Rotter [22] provides a useful tool for measuring individual differences to the extent to which reinforcement is viewed as a consequence of one's own behaviour or a consequence of such forces as chance, fate or powerful others. Singh [24] found that socio-culturally deprived students were significantly more external and chance oriented as compared to non-deprived students. Significant differences between internal - external locus of control in group conformity behaviour were found by Singh [25]. The results of the study conducted by Rao, et al. [20] suggested that the relationship between locus of control orientation and adjustment was not very clear. Locus of control orientation was found to determine the use of specific coping behaviours, but did not play a significant role in determining the experience and perception of stressful life events. Several studies have been carried out on religiosity, locus of control and other related variables (see, for instance, [1], [4], [9], [12], [14], [15], [23], [29] and [33]. The present paper reports data pertaining to the religiosity and locus of control among male and female college students.

Hypotheses :

(i) There would be significant difference between male and female students in terms of religiosity.
(ii) There would be significant difference between male and female students in terms of locus of control.

Sample :

The sample comprises of 100 male and 100 female students studying in class XI and XII in Hardwar District. The sample was selected by incidental sampling technique.

Scales :

Bhushan's Religiosity scale [2] and Rotter's Locus of control scale as standardized by Kumar and Srivastava [13] were used to measure the two variables of religiosity and locus of control among college students.

RESULTS AND DISCUSSION

The results confirm the contention of the hypothesis of significant difference between male and female students on religiosity scores (Table 1). The female students with a mean of 128.39 are found to be more religious in comparison to male students with a mean score of 121.62. This difference may be attributed possibly to cultural training and child rearing practices in the society. These appears to be a different role expectations in regard to boys and girls. Besides the temperamental difference also attracts girls towards religion. Similar conclusions may be found in the investigations conducted by Helode and Barlinge [11], and Bhushan and Singh [4].

Table - 1

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Religiosity

Students	N	Religiosity Scores		
		Mean	S.D.	t-value
Male	100	121.62	17.34	2.82
Female	100	128.39	16.58	$P < 0.01$

Table - 2

Showing Mean and Standard Deviatioin of Male and Female Students in Terms of Locus of Control.

Students		Locus of control scores		
	N	Mean	S.D.	t-value
Male	100	9.96	3.13	3.40
Female	100	11.56	3.43	$P < 0.01$

The results in regard to locus of control confirm that the female students are more external in their locus of control orientation than boys (Table 2). Female students are found to have developed a high "expectancy of external control". They believe that the events in their lives are controlled by such factors as luck, chance and super natural powers which are beyond there control. On the other hand the male students have greater faith in their own abilities and believe that their own abilities and efforts determine and shape their future.

The genesis of this difference can be directly traced in the amount of faith which boys and girls have in religion as discussed earlier, religion is a "faith in a power beyond himself; Since girls are more religious they have greater faith in super natural powers.

These findings are in line with other studies reported by Person and Schneider [18], Blass [5] and Chandran [6] and Kunhikrishnan and Mathew [12] on sex differences in Internal-External locus of control. Results of Helode and Barlinge [11] indicate that externality and religiosity are positively correlated. Females are more religious than males. Molinari and Khanna [17] have suggested that adopting an external orientation when situations are uncontrollable as a defense against negative self-evaluation is a positive characteristic. Such persons attribute failure to luck as a defensive manoeuvre, and yet, resemble internals on behavioural indices of task performance.

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ON THE SOLUTIONS OF VISCOUS INCOMPRESSIBLE FLUID FLOW

N. Yadav*

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ABSTRACT

A technique has been developed to combine two or more known solutions of uniform flow of viscous - incompressible fluid in pipes to a single new solution. Also in the light of the descussion a very bold proposition in the form of a principle has been put forward. This principle has been named as ***the principle of second degree harmonics***. The complete study has been divided into two sections. Section I deals with the formation of a new technique to combine two or more known solutions of viscous - incompressible fluid - flow to a single one and its applications. Section II deals with a particular case, how to derive a solution for a regular figure of *n*-sides from the solution for an equilateral triangle.

Mathematics Subject Classifications (1991): 76

Keywords and Phrases: Compoundable, viscous - incompressible fluid, Boussincsq's' formula.

SECTION I

In a problem of uniform flow of viscous - incompressible fluid in pipes, the boundary along which the velocity vanishes is a closed figure. Any section of this closed figure by a chord, we shall call a segment. Every chord divides the figure into two segments.

Let us consider two such segments taken from two different solutions and they are named as *A* and *B*, the bounding chord being *PQ* and *RS* (fig. 1).

We shall enquire into necessary and sufficient conditions under which *A* and *B* may be règarded as two segments of the same pipe so that by combining the two segments we may arrive at a new solution. The conditions must be :

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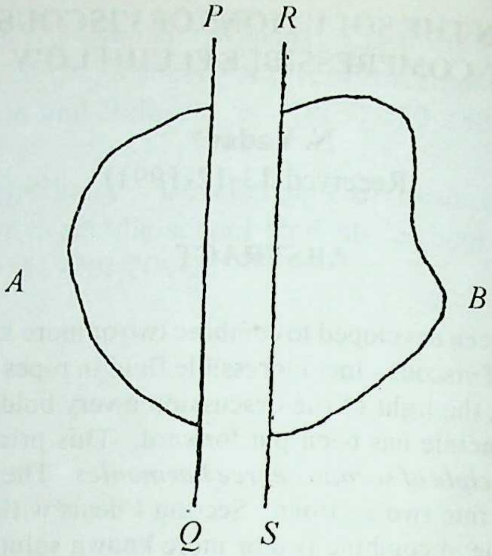

Fig. 1

Geometrical Ccondition :

Chord PQ must be equal to RS

Mechanical Ccondition :

The velocity function in A at every point of PQ must have the same value as the velocity function in B at the corresponding point in RS. By correspondence we mean the coincidence in superposition of PQ over RS.

The mechanical condition guarantees that along the planes of PQ and RS the flow will be the same at every point, so the flow on either side of this common plane will not be disturbed.

Two segments satisfying the geometrical and mechanical conditions, we shall call *compoundable*.

Segments of two known solutions may be joined to give a new solution if they are compoundable. A segment is always compoundable to its mirror image for if D is the mirror image of C as shown in Fig. (2), then obviously the geometrical and mechanical conditions are satisfied.

Now we proceed to consider instead of segments the sector bounded by

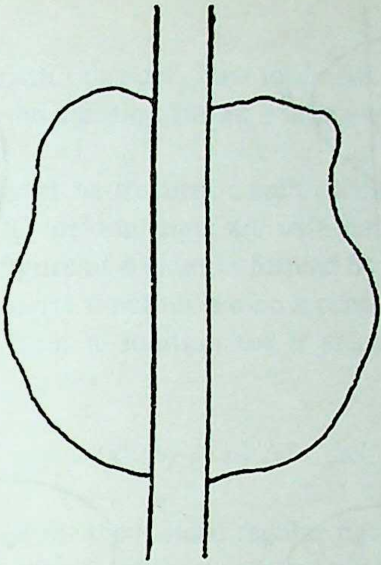

Fig. 2

radii vector making an angle $2\pi/n$, n being an integer. If n sectors be placed in such a way that each is the mirror image of adjoining one, the vertices coinciding, then the geometrical and mechanical conditions for compoundability are satisfied. Only if n is odd the two radii vector must be of equal length. When n is even they may be of unequal length.

We shall now consider some applications of the technique developed above. Considering the segments of an ellipse. Let A be the major segment containing the point of maximum velocity and B, a minor segment (Fig. 3). By combining each with its mirror image we arrive at two solutions L and M (Fig. 4 & 5) respectively. M is specially notable for its peculiarity of having two points of maximum velocity. By taking a major segment of M as indicated by dotted lines, we may get a solution with four points of maximum velocity as shown in Fig. 6. There is no limit to the variety of solutions that can be derived from one known solution. Next, let us consider a sector bounded by radii vectors making angle $2\pi/n$ n being an integer. Taking again the ellipse $PLQN$, let us form two sectors, one major sector containing the centre 0, and the other a minor sector, by two equal intersecting chords PQ and LN, intersecting at a point M on the major axis (Fig. 7).

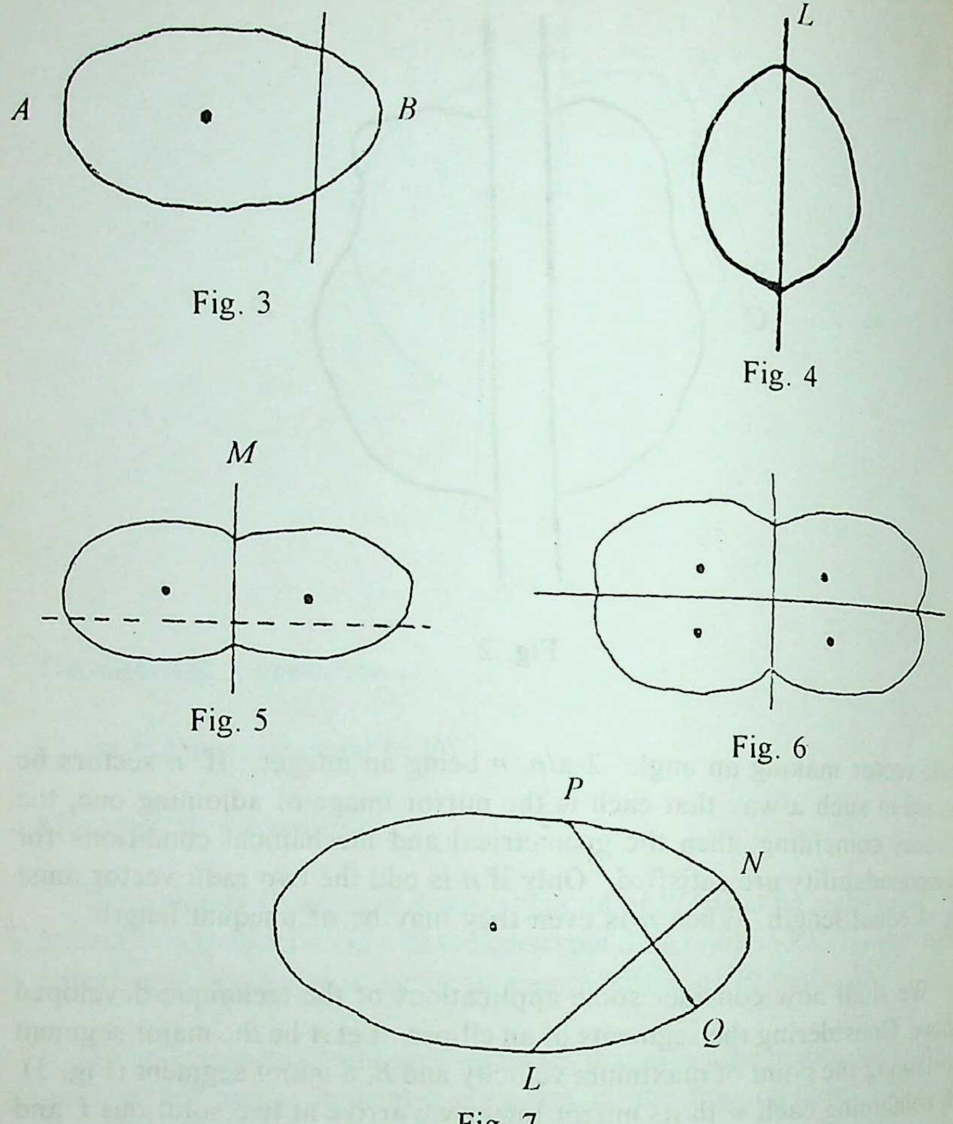

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 3, 4, 5, 6 & 7

The minor sector repeated *n* times gives a solution when the boundary consists of *n* arcs of an ellipse. In this solution there is one point of maximum velocity. The major sector repeated *n* times is more remarkable as it gives a solution with *n* points of maximum velocity.

In all such cases the area, the total flow per unit time, average velocity, maximum velocity, *k* and *k′*, are calculated by considering a single segment or sector and multiplying by the necessary integer.

SECTION II

In this section we shall consider, how to derive a solution for regular figure of n sides, from the solution for an equilateral triangle.

The best method would be to form a sector with the centre O as vertex and radii vectors OP, OQ making angle π/n with the central axis as shown in Fig. 1. The regular figure of n sides is formed by repeating the sector n times round O. The velocity function is also repeated in the same manner. In this way we shall form a solution for n sided regular figure from Boussinesq's formula,

$$w = -(K/4b)\,(y-b)\,(y^2 - 3x^2)$$

The velocity function for the n sided regular figure will be obtained by repeating the above velocity function n times round O' as shown in Fig. 2.

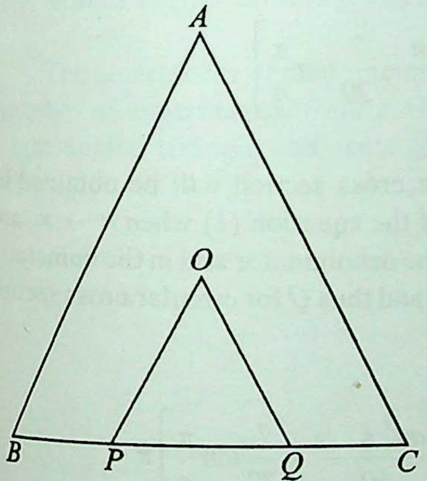

Fig. 1

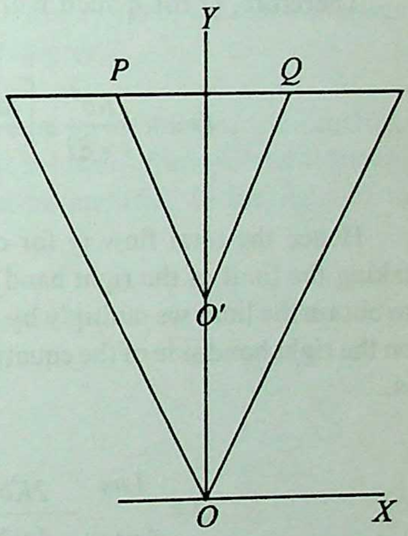

Fig. 2

Fig. 1 & 2

We shall calculate the total flow in the sector $PO'Q$ and thus get the total flow for the regular n sided figure.

For the sector $PO'Q$, total flow Q is given by,

$$2\int_{\theta=0}^{\theta=\frac{\pi}{n}}\int_{r=0}^{r=\frac{b}{3\cos\theta\, k}} -\left(K/_{4b}\right)[r^3\cos^3\theta - 3r^3\cos\theta\sin^2\theta + br^2\cos^2\theta +$$

$$br^2\cos^2\theta + br^2\sin^2\theta - 4b^3/_{27}]\, r\,dr\,d\theta$$

$$2\int_0^{\pi/n} -\frac{Kb^4}{4\times 81}\left[\frac{1}{20\cos^4\theta} - \frac{2}{5\cos^2\theta}\right]d\theta$$

$$Q = -\frac{Kb^4}{4\times 81}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Therefore, Q for n such will be given by

$$Q = -\frac{2Kb^4}{4\times 81}n\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]$$

Hence the total flow Q for circular cross section will be obtained by taking the limit of the right hand side of the equation (1) when $n \to \infty$ and to obtain the limit we multiply by π in the denominator and in the numerator on the right hand side of the equation (1) and thus Q for circular cross section is,

$$Q = \lim_{n\to\infty} -\frac{2Kb^4}{4\times 81}\frac{\pi}{n}\left[\frac{\tan^3\pi/n}{60} - \frac{7}{20}\tan\frac{\pi}{n}\right]\pi$$

$$Q = \frac{2Kb^4}{4\times 81}\cdot\frac{7}{20}\pi = \frac{kb^4}{\frac{40}{7}}\pi$$

Since $a = b/3$; a will be the radius of the circular cross section. We thus see that the total flow in tube of circular cross section derived from *Boussinesq's* formula for equilateral triangle does not come equal to the usual value, but is *greater* in the ratio 7:5.

It has been established by the author in his paper' *A generalized study of a viscous incompressible fluid flow through various cross sections of a tube*' that the formula for the total flow in circular tube, when derived from the formula for rectilinear figure, agrees with the usual value. One thing of remarkable importance is that both the usual velocity function for circular cross section and our formula for rectilinear figures, contain harmonics of second degree only. *Boussinesq's* formula, however, contains harmonics of third degree.

There does not seem to be any flaw in the discussion at section (I), so we are tempted to put forward a very bold proposition in the form of a principle. This principle we shall call the *principle of second degree harmonics*. The principle may be stated in the following words.

In all cases of Laminar flow of viscous incompressible fluids through tubes of cross section of any shape the velocity function can not contain Harmonics higher than those of second degree.

The correctness of the principle has yet to be established from sufficient number of experimental results. Thus the solutions for equilateral triangle, right angled triangle and rectangles can be modified in the light of this principle.

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VEDIC GEOMETRY*

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(Received 23-12-1991)

ABSTRACT

This talk intends to discuss *Śulbas'* period, and give some glimpses from *Brāhmaṇas* and *Śulbas'* concerning the origin of Pythagorian triplets and the irrationality concept of real numbers.

Mathematics Subject Classifications (1991) : O1A32

Keywords and Phrases: *Śulba Sūtra*, *trita*, pythagorian triplets

INTRODUCTION

According to the *Jaina* canon, geometry is the lotus of Mathematics (see *Sūtrakrta* (II. 1. 154), cf. (10), p. XII). Ancient Indian savants are now credited for fundamental pioneering work in geometry A very deep analysis by Seidenberg (See [21], p. 108) concludes "... Greek geometry (especially the Theorem of Pythagoras) did not somehow make its way into *Vedic* geometry, as Greek geometry is only supposed to have started about 600 B.C.".

First I quote the following *mantras* from *Yajurveda* (see [17], p. 210):

एको॑ च मे ति॒स्रश्च॑ मे ति॒स्रश्च॑ मे पञ्च॑ च मे॒ पञ्च॑ च म स॒प्त च मे स॒प्त च मे॒ नव॑
च मे॒ नव॑ च म॒ऽएका॑दश च म॒ऽएका॑दश च मे॒ त्रयो॑दश च मे॒ त्रयो॑दश च मे॒ पञ्च॑दश
च मे॒ पञ्च॑दश च मे स॒प्तद॑श च मे स॒प्तद॑श च मे॒ नव॑दश च मे॒ नव॑दश च
म॒ऽ एक॑विँ॒शतिश्च म॒ऽ में त्रयो॑विँ॒शतिश्च मे॒ त्रयो॑विँ॒शतिश्च मे॒ पञ्च॑विँ॒शतिश्च
मे॒ पञ्च॑विँ॒शतिश्च मे स॒प्तविँ॒॑शतिश्च मे स॒प्तविँ॒॑शतिश्च मे॒ नव॑विँ॒शतिश्च मे॒
नव॑विँ॒शतिश्च म॒ऽएक॑त्रिँ॒शच्च म॒ऽएक॑त्रिँ॒शच्च मे॒ त्रय॑स्त्रि ꣳशच्च मे य॒शेन॑
कल्पन्ताम् ॥१ ॥

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*Talk presented at National Seminar on Science in Ancient India, Kumaun Univ., Nainital, Oct. 12-16, 1991.

च त॑ स्रश्च मे॒ऽष्टौ च मे॒ऽष्टौ च॑मे॒ द्वादश च मे॒ द्वाद॑श च मे॒ षोड॑श च मे॒
षोड॑श च मे॒ विँ॒शतिश्च॑ मे॒ विँ॒शतिश्च॑ मे॒ चतु॑र्विँ॒शतिश्च मे॒ चतुर्विँ॒शतिश्च
मे॒ऽष्टाविँ॒शतिश्च मे॒ द्वात्रिँ॒शच्च मे॒ द्वात्रिँ॒शच्च षट्त्रिँ॒शच्च मे॒ षट्त्रिँ॒शच्च मे॒
चत्वारि॒ँ॒शच्च मे॒ चत्वारिँ॒शच्च मे॒ चतु॑श्चत्वारिँ॒शच्च मे॒ चतु॑श्चत्वारिँ॒शच्च
मेऽष्टाच॑त्वारिँ॒शच्च मे॒ यज्ञेन॑ कल्पन्ताम् ॥२ ॥

य० अ० १८ । मं० २४,२५ ॥

The first *mantra* means 1, 3, 5, 7, etc. It also means 1, 3^2, 5^2, 7^2, ... The second *mantra* means 4, 8, 12, 16 etc. For several such interpretations, refer to [3]. However, *Svāmī Dayānand* [17] says that several techniques of arithmetic may be derived from these *mantras*, and similarly fundamentals of algebra and geometry may be obtained from *Vedas*.

The following *mantra* from *Rig Veda* (see [17], p. 212) or [3, p. 326] talks about measuring instrument, circumference etc. :

कासी॑त् प्र॒मा प्र॑तिमा किं नि॒दानमाज्यं॒ किमा॑सीत् परि॒धिः क आ॑सीत् ।
छन्द॒ः किमा॑सी॒त् प्रउ॑गं॒ किमु॒क्थं यद्दे॒वा दे॒वमय॑जन्त॒ विश्वे॑ ॥४ ॥
ऋ० अ० ८ । अ० ७ । व० १८ । मं० ३ ॥

References of circle and its circumference, triangle and polygons are available in *Rig Veda* (see [3], pages 326-327).

TRITAH

Surprisingly Sanskrit name of π is *Trita*. In *Atharva Veda*, the ratio of circumference of a circle to its diameter is called *trita*, which is approximately 3:1 (see [3], p. 326).

According to Wendy O'Flaherty : "The riddles in the *Rig Veda* ... do not have, nor are they meant to have, answers. They are not merely rhetorical, but are designed to present one half of a Socratic dialogue through which the reader becomes aware of the inadequacy of his certain knowledge". I wish to say that neither it is, nor it can be the intention of the *mantra* to give a good approximation of *trita*. It is enough that the intended meaning of *trita* is not a rational number implicitly. There are several references in *Mahābhārata* which imply that the *trita* is 3. See [4] for a detailed analysis of *trita* or π in *Mhābhārata*, and [6] for a detailed analysis of the value of *trita* or π available in ancient Indian mathematical compositions. It may be mentioned that *Āryabhata's Āryabhatiya* (499 A.D.)., which is the earliest mathematical text that bears the name of the author, gives a good approximation of π (see also [1], [2], [7], [9], [10], [12], [23] - [25]).

VEDIC PERIOD

A section of *Vaidikas* hold *Veda* as beginningless (indeed as old as this creation, (see [17], pp. 12-37) and *Svayambhu*. However, the *savites* hold that *Vāgisā Siva* set forth the *Vedas*, the six *aṅgas* and *āgamas*. According to seventh century Tamil *Saivite saint Appar, Lord Siva* is the embodiment of the four *Vedas* and six *aṅgas* (see, for instance, [13], Chap. 6). However, following Keith, and Satya Prakash (see [15], pp. 5-6; and also Renfrew [16] and Datta Singh [2], Part I, pp. 1-2), *Vedas*, the earliest extant compositions, and the subsequent *Vedic* literature are assigned the following dates :

Vedas	3000 B.C.
Brāhmaṇas	2500 - 2000 B.C.
Baudhāyana Sūtra	800 B.C.
Āpastambha Sūtra	600 B.C.
Manava Sūtra	550 B.C.
Pāṇini Sūtra	500 B.C.
Kātyāyana Sulba Sūtra	200 B.C.

Two recent investigations push back the above mentioned date of *Vedas* significantly. The recent deciphering of Harappan language by Louisiana State University Professor and Cryptologist, S. Kak suggests *Vedic* period around 8000 B.C. A recent deep analysis under the title "*Mahayuga*: the great cosmic cycle and the date of the *Veda*" concludes the *Ṛg Veda* is at least as old as 7300 B.C. If we accept 8000 B.C. for *Ṛg Veda*, the other dates of *Brāhmaṇas* and *Sūlba Sūtras* should also be pushed back proportionately.

THEOREM ON RECTANGLES

Pythagoras theorem (also called *Baudhāyana - Pythagoras* theorem in some recent High School text books) is available implicitly in *Brāhmanas* and explicitly in *Śūlba Sūtras* which mewans "*Sūtras of cord*". For a detailed analysis refer to Seidenberg [20] - [21] (see also [2], [11], [12], [14], [15], [24] and [25]).

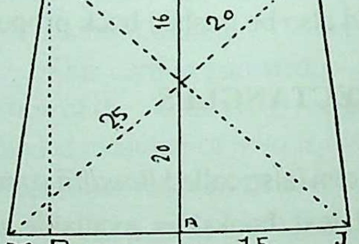
Consider two sets of *mantras* from *Śatapatha Brāhmaṇa*.

Set I :

तद्य एष पूर्वाध्यों वर्षिष्ठ स्थूणाराजो भवति । तस्मात्प्राङ् प्रक्रामति त्रीन्विक्रमां
रतच्छङ्कुं निहन्ति सोऽन्तः पातः ॥१ ॥
तस्मान्मध्यमाच्छङ्कोः । दक्षिणा पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति
सा दक्षिणा श्रोणिः ॥२ ॥
तस्मान्मध्यमाच्छङ्कोः । उदङ् पञ्चदश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति
सोत्तरा श्रोणिः ॥३ ॥
तस्मान्मध्यमाच्छङ्कोः । प्राङ् पट्त्रि ꣳशतं विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति
स पूर्वार्धः ॥४ ॥
तस्मान्मध्यमाच्छङ्कोः । दक्षिणा द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति स
दक्षिणोऽ ꣳ सः ॥५ ॥
तस्मान्मध्यमाच्छङ्कोः । उदङ् द्वादश विक्रमान्प्रक्रामति तच्छङ्कुं निहन्ति
स उत्तरोऽꣳस स एषा मात्रा वेदेः ॥६ ॥

अध्याय- ५ ब्राह्मण १

This means : From the tallest pole in the *s'ā lā* move 3 *padas* (foot) towards east, and fix up a nail there. Call it "*antah pā ta*". From the centre of this nail move 15 *padas* towards south, and fix up a nail there, which is called south (right) pelvic. From the centre of the nail move 15 *padas* towards north, and fix up a nail there, which is called north (left) pelvic. From the centre of the nail move 36 *padas* towards east, and fix up a nail there, which is called *Purvardha* (half-east point). From the centre of (this) nail move 12 *padas* towards north, and fix up a nail there which is called south (right) pectoral. From the centre of the nail move 12 *padas* towards north, and fix up a nail there, which is called north (left) pectoral. These are the *mā trā s* (magnitudes) of the (*Mahā*) *vedi*.



Evidently, *Mahāvedī* is *JKLM* which is an isosceles trapezoid. *P* is the starting point and *A* = *antah pā ta* where *PAE* is a line from west to east. Subsequent *mantras* give reasons for the choice of the lengths *MJ*, *LK* and *AE*. The vedi is compared with a woman whose lower part is wider than the upper part.

since a woman is considered superior because of her womb which is responsible for creation (see [26], pages 429-431). Note that the *vedī* is loaded with Pythagorian triplets such as (12, 36, 39), (27, 36, 45).

Set II:

(ꣳ) स॒ व्वेद्यन्तात् । (त्ष॒) ष॒ट्त्रिꣳ शत्प्रक्रमां प्रा॒चीं व्वे॒दिं व्वि॒भिमीते त्रिꣳ श॒तं पश्चा॒त्तिर॒श्चीं च॒तुर्व्विꣳ शति पु॒रस्ता॒त्तन्नवति॒ः सैषा॒नवति॒प्रक्रमा व्वे॒दिस्त॒स्याꣳ सप्त॒विधमग्निं व्वि॒दधाति ॥४ ॥

त॒दाहुः । क॒थमेष॒ सप्त॒व्विध ऽएत॒या व्वे॒द्या स॒म्पद्यतऽइ॒ति द॒श वा॒ऽइमे पु॒रुषे प्राणा॒श्चत्वार्स॒ङ्गन्न्यात्मा॒ पञ्चदश॒ ऽएव॒ं द्विती॒यऽएव॒ं तृ॒तीये पट्सु॒ पुरुषेषु नवतिर॒थैकः पु॒रुषो॒ऽत्येति पा॒ङ्क्तो वै पु॒रुषो लो॒म त्व॒ङ्माꣳ स॒मस्थि मज्जा पा॒ङ्क्तोऽइयं व्वे॒दिश्चतस्रो दि॒श ऽआत्मा॒ पञ्च॒म्येव॒मेष॒ सप्त॒विध ऽएत॒मा व्वे॒द्या स॒म्पद्यते ॥५ ॥

These *mantras* also describe the *Mahā vedī JKLM* (cf. [18], p. 37) but there is no mention of pole P (and hence of PA). However, there is one significant aspect concerning the measurements. In fact, it says that this is 90 *padas vedi*, i.e., a *vedi* of 90 units/feet. Note that $JM + AE + KL = 30 + 36 + 24 = 90$ and $JL + MK = 45 + 45 = 90$. Note also the triangle BLJ is a right triangle and $27^2 + 36^2 = 45^2$, i.e., $JB^2 + BL^2 = JL^2$. Seidenberg [21, p. 106] has rightly said confidently that : "I therefore regard it as certain that the *Śatapatha Brahmaṇa* knows the Theorem (of Pythagoras)". He however has not mentioned the 90-*pada - vedi* aspect.

SAVIS'ESA

Śulbas, the manual for construction of altars have different approximation for √2. This suggests an understanding of difference between raional and irrational numbers, since *Vedic* priests knew that their (approximate) value of √2 was not exact (see [8, p. 43] and [20, p. 328]). In *Śulbas*, if one wishes to turn a circle into a square (see, for instance, Baudhāyana *Śulbas* I.60; *Āpastamba Śulbas* III.9; *Kātyāyana Śulbas* III.14), then one comes across a notable contribution of *Sulbas*, viz., a rational approximation of √2, which is generally termed as *Savis'esa*. The *Śulbas* give

(V) $2 = 1 + 1/3 + 1/3.4 - 1/3.4.34$

It may be mentioned that the commentator *Rāma* of *Naimisa* (near Lucknow) gave the following improved approximation (see Kapur [9, p. 85] and Khadilkar [11, p. 83]).

(R) $\sqrt{2} = 1+1/3 + 1\backslash 3.4 - 1/3.4.34 - 1/3.4.34.33 + 1/3.4.34.34$

Several proofs of (V) have been suggested by G. Thibout (cf. [14] and S.D. Khadilkar [11] (see also [9, p. 98]). Simple arithmetical and algebraic proofs may be given to (V) and (R). The latter may further be improved as follows :

(S) $\sqrt{2} = 1+1/3 + 1/3.4 - 1/3.4.3.4 - 1/3.4.34.33 + 1/3.4.34.34$
$+ 1/3.4.34.33.34.36$
(Herein 36 may also be replaced by 35)

The approximation (V), (R) and (S) are correct to 5, 7 and 8 places of decimals respectively.

The traditional Indian or *Vedic* mathematics from *Āryabhata* (476 A.D.) to *Mādhava* (1340 - 1425 A.D.) embodies many surprising and beautiful concepts and results in algebra, geometry, trigonometry, mensuration, theory of numbers, the basic concept of limit, calculus (differential and integral both) and Taylor's series for *sines* and *cosines* upto second powers. However, most studies of Indian Mathematics have unfortunately (in fact unfortunate to historians and scientific community) not been able to see its contributions to the scientific methods. Thanks to the West that its modern savants were the first to bring an appreciation of a portion of huge Indian mathematical and astronomical compositions and the origin of geometry into the notice of the scientific community. I trust there is still a lot to be investigated, and hope that the era of neglect is perhaps over.

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STUDIES ON POLLUTION REDUCTION POTENTIAL OF EICHHORNIA CRASSIPES GROWN IN INDUSTRIAL EFFLUENT

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ABSTRACT

The present investigation was conducted to find out the pollution reduction efficiency of *Eichhornia crassipes* grown in industrial wasts of Indian drug pharmaceutical limited, Virbhadra. Results obtained during this investigation showed that plant growth was healthy, leaf numbers were increased four times within 30 days and was better than that of the plants grown in tap water (control IInd). There was considerable increase in D.O. level i.e. 1.5 (mg/lt) to 3.75 (mg.It) and BOD value was decreased i.e. 240.6 to 175.0 (mg/lt). But at the same time this value was reversed where *Eichhornia* plants were grown in tap water i.e. control IInd. A declined trend was recorded in other parameters such as hardness, alkalinity and chloride etc.

Keywords : ***Eichhornia***. Pollution, D.O. BOD effluent.

INTRODUCTION

It is a well known fact that aquatic bodies are more susceptable for contamination due to various type of waste discharged into these. These waste includes industrial effluent, sewage and domestic water etc. Raw sewage contains various organic and inorganic materials such as cellulose, carbohydrates, fats protein urea and amines etc. Direct discharged of these polluted water often creat health problems in densly populated area of the country. Large number of waste are being discharged into aquatic bodies. These aquatic bodies contain smaller or larger number of weed dependig on nature of aquatic body. Generally aquatic weeds are still regarded by majority as "menace" and nuisance because they are not yet aware of the great potential and economic values of these profusely growing uncontrollable plants.

Eichhornia crassipes is one of the most abundantly fast growing weed in tropical and subtropical parts of the world. This fast growing troublesome weed is a good source of pulp, paper and human food. Mazid (6) reported some tribal in Bangladesh eat the flowers, young peduncle, and young petiols of this plant.

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Aquatic weeds are good source of protein as such and these may be substitute for the diet harvivorus live stock. *Water-hyacinthis* also consumed by cattle but for good consuption it has to be mixed with right proportion of other feed i.e. 30%. Cattle feed consisting equal proportion of this weed with 5% molases supported good growth of cattle (Mazid (6). This contains protein from 9.14 to 14.37/dry weight. Mazid (5) reported that 672.00, 103.74 and 84.64 lb protein is produced monthly per acre by *E. crassipes, Salvinia* sps and *Azolla pinnata* respectively in Bangla desh.

Water hyacinth can also be used as manure. Compost prepared from aquatic weeds are more effective than cow dung compost for yield in several crops. Twenty two, 25.5 and 7.5 percent more yield was gained by using compost of *E. crassipes,* than cow dung manure in Rice. Onion and Turnip respectively Mazid (6). Besides these they are used in Sudan for bio-gas production. According to NASA. Water-hyacinth harvested from one hectare will produce more than 70,000 m3 of Bio-gas.

The major role of these aquatic weeds in aquatic body to reduce the pollutants. It has been reported by National Academy of Sciences U.S.A. that some aquatic weed can scavenage inorganic and some organic compounds from water. These weeds are capable of absorbing and incorporating the dissolved materials into their own structure, Reddy and Debusk (9) reported that eight aquatic macrophytes including *E. crassipes* can remove the N and P from nutrient enriched water.

Water hyacinth is a natural resource to reduce pollution load from waste water and can be utilized for treatment of waste water. Important contribution in these lines has been made by (Schultze, (1966), Scheffield, (11); Sinha and Sinha, (12); Miner, (7); Rao et al. (8). Dunigon et al. (4); Walverton, (14); Walverton and Donald, (13)

Like other aquatic bodies, Ganga also receives various kinds of pollutants throughout its course. Albiet water pollution board in India was framed in 1974 but due to several problems it could not be so effective.

Keeping in view of the economy of the country and nutritional value as well as agricultural importance of the aquatic weed it was thought to make a preliminary study on pollution reduction potential of *E. crassipes* of larger leaves and their growth on industrial effluent under Indian environment.

MATERIALS AND METHODS

Plants of *Eichhornia crassipes* of larger leaves (Water hyacinth) used in the present investigation were collected from the local ponds. Industrial effluent was collected from Indian drug pharmaceutical limited (IDPL), a drug industry, located at Virbhadra Rishikesh, India. Fourty lts. effluent was filled in each cemented tanks. Pure eflluent (without dilution) was used for the experiment. Controls were of two types, control first was pure effluent without plants and control second was tap water with *Eichhornia* plants. Three replicate were used for each set.

Four plants of *Water hyacinth* were kept in each tank at begning of the experiment. Each plant contained seven leaves. Various physico chemical and biological characters of the effluent were evaluated *at* initial stage of the experiment. Analysis of the effluent was done as suggested by APHA [1]. A regular analysis of the effluent and observation on growth of the plants was made at interval of 15 days for the period of one month. The experiment was reconducted to confirm the findings.

RESULTS AND DISCUSSION

Results obtained during this investigation are given in the Table. In general there was no change in physico chemical characterestics of first control (Pure effluent without plants). However negligible reduction in alkalinity, hardness and BOD was recorded. Markedly reduction in various parameters was noted in pure effluent where water hyacinth plants were grown. Enhancement of dissolved oxygen from 1.5 to 3.75 (mg/lt) was found after 30 days interval. Profuse growth of the plants was found in pure effluent siamilar to the plants grown in tap water. Number of leaves were increased four times after one month. (Table).

It is very much obvious that *Eichhornia crassipes* of larger leaves is capable to reduce the polutants from water as it is evident with enhacement of dissolved oxygen in the effluent (Table). Albeit BOD was decreased gradually and dissolved oxygen was increased in the effluent but BOD was slightly increased in second control. Sixty percent reduction in BOD and suspended particles was recorded by Walverton etal. [14] when sewage was passed from wsater hyacinth plants. Further Walverton and Donald [13] reported that one acre of dens growth of waterhyacinth can treat waste water generated from 2000 people. Dinges [3], a scientist of NASA, demonstrated that this weed based filtration system can function effectively and such treatment plants for waste from municipal are under trial operation.

Increased in BOD in second control (Tap water with plant) is similar with findings of Macvea and Boyd [2]. They reported that water hyacinth decreased dissolved oxygen when it is grown in clean water. This is a peculiar feature of this plant that it reduces pollution load when grown in waste water but creates pollution when grown in clean water.

Four times increse of leaves after 30 days on effluent showes its rapid and profuse growth. This indicates that water hyacinth can grown well in effluent of such industries.

With the above findings and researches done by workers, on utility of this weed for waste treatment and in agriculture, it is desirable that this can be clultivated on stored waste water. After a definite time weed should be harvested regularly for compost preparation and cattle feed Mazid [5]. This will help in multiways at one time.

Table-IV : Effect of Eichhornia Plants on I.D.P.L. effluent.

Sl. No.	Experimental tanks	No of days	No of Plants	No of Leaves	pH	Chloride	Hard-ness	Alkali-nity	Conduc-tivity	D.O.	B.O.D.
1.	Effluents 100% with plants	Q	4.00	7.00	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C 1 2.	Effluents 100%	0	-	-	6.20	15.00	95.00	89.00	0.230	1.50	240.60
C11 3.	Tapwter with plants	0	4.00	7.00	7.01	10.00	175.00	162.00	0.218	8.00	0.88
1.	Effluents 100% with plants	15	4.00	16.50	5.90	5.00	70.00	80.00	0.235	2.30	210.00
C1. 2.	Effluents 100%	15	-	-	6.00	15.00	95.00	84.00	0.228	1.00	238.00
C11.3.	Tapwater with plants	15	4.00	16.50	6.80	5.00	150.00	153.00	0.220	9.80	1.40
1.	Effluents 100% with plants	30	4.00	31.50	5.80	5.00	70.00	75.00	0.230	3.75	175.00
CI 2.	Effluents 100%	30	-	-	6.00	15.00	95.00	84.00	0.223	0.40	232.00
CII 3.	Tapwater with plants	30	4.00	30.60	6.70	-	150.00	148.00	-	10.40	2.50

Control Ist (Pure without effluent plants)
Control IInd (Tap water with plant)

Developing countries like India should use such kinds of weeds (biological means) for treatment of waste water in the pattern of NASA. This would require proper weed management. A large number of aquatic weeds including waterhyacinth can be tested for treatment of municipal waste and compost preparation as well as bio gas production etc.

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A NOTE ON THE MODULAR SEMI-GROUP RING OF A FINITE IDEMPOTENT SEMIGROUP

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In [1] Johnson has proved when G is a group of prime power order p and K a field having only p-elements then G^*, the mod-p-envelope of G has a group structure. In this note we prove for an idempotent semi-group S of finite order having an identity I and K a field of characteristic p, S^* the mod p-envelope of S has a monoid structure, with non-trivial idempotents in it. For more properties about modular group rings please refer [1] and [2]. Just we define a semi-group ring of a semi-group S over a ring R analogous to group rings as follows:

Definition 1.
Let R be a ring (not necessarily commutative and not necessarily with a unity) and let S be a semi-group. We define the semi-group ring $R[S]$, to be the ring consisting of all finite formal sums $\Sigma\alpha_i x_i$ ($\alpha_i \in R$; $x_i \in S$) with the obvious definition of addition and with multiplication induced by the given multiplication in R and S according to the rule.$(\Sigma\alpha_i x_i)(\Sigma\beta_j Y_j) = \Sigma(\alpha_i\beta_i) x_i y_j$ ($\alpha_i, \beta_j \in R$; $x_i, y_j \in S$). We take $(\Sigma\alpha_i x_i)$ to mean $\Sigma(\alpha_i) x_i$ ($\alpha_i \in R$; $x_i \in S$). If R has a unity I_R then we impose the further condition $I_R x = x$ for all $x \in S$. In particular $R[S]$ can be viewed as a free left R-module. When R is a commutative field and S a monoid $R[S]$ is a semi- group algebra. Throughout this paper by S we mean a commutative idempotent semi-group under multiplication with multiplicative identity 1.

Example 1.
Let $K = (0, 1)$ and $S = \{1, g_1, g_2 \mid g_1^2 = g_1, g_2^2 = g_2, g_1 g_2 = g_2 g_1\}$ be a semigroup. KS the semigroup ring of S over K is given by $KS = \{0, 1, g_1, g_2, g_1g_2, 1+g_1, 1+g_2, 1+g_1 g_2, g_1 + g_2, g_1 + g_1 g_2, g_2 + g_1 g_2, 1+ g_1 + g_2, 1+ g_1 + g_1 g_2, 1+ g_2 + g_1 g_2, g_1 + g_2 + g_1 g_2. 1 + g_1 + g_2 + g_1 g_2\}$. $S^* = 1 + U$ where $U = \{\Sigma\alpha_i g_i \mid \Sigma\alpha_i = 0\}$ ($\alpha_i \in K$, $g_i \in S$); $S^* = 1 + U = 1 + \{g_1 + g_2, 1+g_1, 1+g_2, 1+g_1 g_2, g_1 + g_1 g_2, g_2 + g_1 g_2, 1 + g_1 + g_2 + g_1 g_2, 0\} = \{g_1, g_2, g_1 g_2, 1 + g_1 + g_2, 1, 1 + g_1 + g_1 g_2, 1 + g_2 + g_1 g_2, g_1 + g_2 + g_1 g_2\}$ clearly S^* *is* an idempotent semi-group with identity and order of S^* is 8.

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Definition 2.
Let S be a senti-group with identity and K a field of characteristic p. KS the semi-group ring of S over K, then S^*, the mod p-envelope of S is $1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K;\ S_i \in S \}$.

Proposition 3.
Le $K = (0,1)$ be the field of integers modulo 2 and $S = \{1, g_i \mid g_i^2 = g_i\, g_i g_j = g_j g_i\}$ be an idempotent semigroup of order n. Then S^* is a monoid in which every element is an idempotent and order of S^* is 2^{n-1}.

Proof.
$S^* = 1+U$ where $U = \{ \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0 \}$.Here $U = \{0$, elements taken two at a time, elements taken four at a time,..., elements taken n at a time if n is even or elements taken $n-1$ at a time if n is odd). Thus order of S^* is 2^{n-1}. This is easy to verify $1 \in S^*$ and every element of S^* is an idempotent as characteristic of K is two.

Example 2.
Let $K = (0, 1,2)$ be the field of integers modulo 3 and $S = \{1, g_1 g_2 \mid g_1^2 = g_1,\ g_2^2 = g_2,\ g_1 g_2 = g_2 g_1\}$ Then S^* is not an idempotent semigroup but contains identity and idempotents.For $S^* = 1 + U = 1 + \{ g_1 + g_2 + g_1 g_2,\ g_1 + 1 + g_1 g_2,\ g_1 + g_2 + 1,\ 1 + g_2 + g_1 g_2,\ 2 + 2 g_1 + 2g_1 g_2,\ 2g_1 + 2g_2 + 2g_1 g_2,\ 2 + 2g_2 + 2g_1 g_2,\ 2 + 2g_1 + 2 g_2,\ g_1 + g_2 + 2 + 2g_1 g_2,\ 2g_1 + 2 g_2, + g_1 g_2 + 1,\ 2g_1 + 2 + g_2 + g_1 g_2,\ 2g_1 + 2g_1 g_2 + g_2 + 1,\ 2g_1 g_2 + 2g_2 + g_1 + 1,\ 2 g_2 + 2 + g_1 g_2 + g_1,\ 2 g_1 + g_2,\ 2g_1 + 1,\ 2g_1 + g_1 g_2,\ g_1 + 2g_2,\ g_1 + 2g_1 g_2,\ g_1 + 2,\ g_2 + 2,\ 2 + g_1 g_2,\ 2g_1 g_2 + 1,\ 2g_1 g_2 + g_2, g_1 g_2 + 2g_2,\ 2g_1 + 1,\ 0\}$ Now S^* contains idempotents and not every element of S^* is an idempotent. $1 \in S^*$ with order of $S^* = 27$.

Proposition 4.
Let $Z_p = (0,\ 1,....,p-1)$ be the field of integers mod-p, p a prime and S be a commutative idempotent semi group of order n with identity 1. Then the following conditions are true in S^*.

(1) S^*, the mod p envelope of S has P^{n-1} elements

(2) $S \subseteq S^*$

(3) S^* has no zero divisors.

Proof.
Since $S^* = 1+U$ where $U = \{ 0, \Sigma\alpha_i s_i \mid \Sigma\alpha_i = 0,\ \alpha_i \in K$ and $s_i \in S\}$ we have order

of S^* to be p^{n-1}. Clearly SCS^*

If $\alpha \in S^*$ be such that there exists a $\beta \in S^*$ with $\alpha\beta = 0$ then we will arrive at a contradiction. Let $\alpha = \Sigma \alpha_i s_i$ and $\beta = \Sigma \beta_j s_j$ with $\Sigma\alpha_i = \Sigma\beta_j = 1$ $\alpha\beta = \Sigma\alpha_i s_i \; \Sigma\beta_j s_j = \Sigma(\alpha_i\beta_i)\, s_i s_j$ if $\alpha\beta = 0$ we must have $\Sigma\alpha_i\beta_j = 0$. Obviously SCS^* as every element of S is contained in S^* (Since $(p-1) + s_i \in U$ for all $s_i \in S$ so $1 + (p-1) + s_i \in S^*$ ie $s_i \in S^*$), S^* has no zero divisors. For if S^* contains zero divisors say α, β such that $\alpha.\beta = 0$; then we must have $\Sigma\alpha_i$ $\Sigma\beta_j = \Sigma\alpha_i\beta_j = 0$ (where $\alpha = \Sigma\alpha_i s_i$ and $\beta = \Sigma\beta_i s_i$) which is impossible as $\Sigma\alpha_i = 1$ and $\Sigma\beta_j = 1$, we have $\Sigma\alpha_i\beta_j = 1$ as both α and β are in S^*. Thus S^* has no zero divisors.

Example 4.
Let $K = (0,1)$ and $S = < s, 1 \mid s^3 = s>$. S^* is a semi group having an idempotent and identity. For $KS = \{0, 1, s, s^2, 1+s, 1+s^2, 1+s+s^2, s+s^2\}$, Now $S^* = 1 + U = 1 + \{0, 1+s, 1+s^2, s+s^2\} = \{1, s, s^2, 1+s+s^2\}$ Now $s^2 \in S^*$ is an idempotent of S^*.

Theorem 5.
Let $K = (0, 1)$ and S be a semigroup with 1 and an element s such that $s^p = s$ where p is any integer. Then S^* is a semigroup with 1 and order of S^* is 2^{p-1}.

Proof obvious

Example 5.
Let $K = (0,1,2)$ and $S = \{1, s \mid s^3 = s)$. The $KS = \{0, 1, 2, s, s^2, 2s, 2s^2, 1+s, 1+s^2, 1+s+s^2, s+s^2, 2+2s, 2+2s^2, 2+2s+2s^2, 2s+2s^2, 1+2s+2s^2, 2+s+2s^2, 2+s^2+2s, 2+s+s^2, 1+s^2+2s, 1+2s^2+s, s+2s^2, 1+2s^2, 1+2s, 2+s^2, 2+s^2, 2+s, 2s+s^2\}$. $S^* = 1+U = 1+\{1+s+s^2, 2+2s+2s^2, 2+s, 0, 2s+s^2, 2+s^2, 1+2s^2, 1+2s, s+2s^2\}$. $S^* = \{1, 2+s+s^2, 2s+2s^2, s, s^2, 2+2s^2, 1+2s+s^2\}$ S^* is a semigroup with one has elements of order two and order of S^* is 9. Thus

Theorem 6. Let $K = (0, 1,..., p-1)$; p a prime and $S = \{s, \bar{1} \mid s^p = s\}$. Then S^* is a semigroup of order p^{p-1} with identity 1.

Proof Obvious.

We are not able to say anything when S is a general semigroup.

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STRESS DISTRIBUTION IN A NON - HOMOGENEOUS CYLINDRICALLY ANISOTROPIC ELASTIC CYLINDER

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ABSTRACT

This paper deals with the determination of the radial displacement and relevant stresses in a cylinder on the basis of three dimensional linear theory of elasticity. The material of the cylinder is orthotropic with cylindrical anisotropy and, in addition, is continuously non-homogeneous, with mechanical properties varying along the radius. The variations of stresses along the radius have been shown graphically.

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Keywords: Orthotropic material, Cylindrical anisotropy, Elastic parameter, Nonhomogeneous material, Hypergeometric function.

INTRODUCTION

The solution for radial deformation and corresponding stresses in a cylindrical shell of homogeneous isotropic elastic media under the action of internal and external pressure was obtained by Lame' as quoted in Love [5].

Since last few decades many investigators have been paying attentions to problems in elasticity in which the material is no longer homogeneous. Due to complications in the governing differential equations these problems are sometimes solved by numerical methods. Grief and Chou [3] have considered such a problem. They have solved the vibration problem of an anisotropic non-homogeneous cylindrical shell without considering any longitudinal extension. Roy [7] has found the radial deformation and stresses in a cylindrically anisotropic non-homogeneous material under the influence of normal pressure on both the boundaries, allowing uniform longitudinal extension. In our present investigation we have considered the problem of stress distribution in an elastic

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cylinder of non-homogeneous material. The medium is assumed to be such that the elastic coefficients at any point vary exponentially with an arbitrary power of the radial distance of the point considered. Due to this assumption the governing differential equation reduces to a confluent hypergeometric equation by suitable transformation. The expression for the radial displacement and stresses in the homogeneous case have been shown to be the limiting values of the corresponding expressions of our results.

FORMULATION OF THE PROBLEM

Taking the axis of anisotropy as the z-axis and using cylindrical coordinate (r, θ, z) the stress-strain relations for cylindrically orthotropic material take the form (cf. Lekhnitskii [4], Armenakas and Ritz [2])

$$\sigma_r = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_\theta = c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}$$

$$\sigma_z = c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz}$$

$$\tau_{\theta z} = c_{44}e_{\theta z}, \quad \tau_{rz} = c_{55}e_{rz}, \quad \tau_{r\theta} = c_{66}e_{r\theta} \tag{1}$$

where $c_{ij} = c_{ji}$, $(i,j = 1,2,3.)$

The strain components in terms of the displacement (u,o,w) are

$$e_{rr} = \partial u/\partial r, \; e_{\theta\theta} = u/r, \; e_{zz} = \partial w/\partial z$$

$$e_{rz} = \partial u/\partial z + \partial w/\partial r, \; e_{r\theta} = e_{\theta z} = 0 \tag{2}$$

Let the nonhomogeneity of the material be characterised by the variations of the elastic parameters c_{ij} according to the following law

$$c_{ij} = \lambda_{ij}\, e^{-kr^m}, \; k > 0 \tag{3}$$

Where λ_{ij}'s are the prescribed parameters of the material concerned and $m(\neq 0)$ is a real number. The case $m = 0$ corresponds to the homogeneous medium.

For the case

$$w = ez \tag{4}$$

where e is a constant and $\partial u/\partial z = 0$, we have a state of plane-strain with a uniform longitudinal extension superposed.

In view of (4) $\tau_{rz} = 0$ and the stress equation of equilibrium in the absence of the body forces reduce to

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{5}$$

Using equations (2), (3) and (4) in (1) we get the stresses in terms of the displacement, then substitution of the stresses in (5) yields the equation of equilibrium as

$$\begin{aligned} r^2 \frac{d^2 u}{dr^2} + (1 - mkr^m) r \frac{du}{dr} - \left(\frac{\lambda_{22} + \lambda_{12} kmr^m}{\lambda_{11}}\right) u \\ = \frac{\lambda_{13}}{\lambda_{11}} emkr^{m+1} - \left(\frac{\lambda_{13} - \lambda_{23}}{\lambda_{11}}\right) er \end{aligned} \tag{6}$$

If we assume that the cylinder $a \le r \le b$ is under the action of internal and external pressures, the boundary conditions may be put as

$$\begin{aligned} \sigma_r &= -p_o \quad \text{at} \quad r = a \\ \sigma_r &= -p_1 \quad \text{at} \quad r = b \end{aligned} \tag{7}$$

Our problem is to solve equation (6) satisfying (7).

Solution of the problem

In view of the transformation

$$z = kr^m, \; u = z^n V, \; n = s/m \tag{8}$$

the equation (6) reduces to

$$z \frac{d^2 V}{dz^2} + (\gamma - z)\frac{dV}{dz} - \alpha V = z^q(\xi z + \eta) \tag{9}$$

where

$$\alpha = (s + \lambda_{12}/\lambda_{11})p$$

$$\gamma = 2n+1$$

$$q = p - n - 1$$

$$s = \surd(\lambda_{22}/\lambda_{11}) = mn$$

$$p = 1/m$$

$$\xi = \lambda_{13}/\lambda_{11}\ epk^{-p}$$

$$\eta = (\lambda_{23} - \lambda_{13})/\lambda_{11}\ ep^2 k^{-p}$$

The complementary equation

$$z\frac{d^2V}{dz^2} + (\gamma - z)\frac{dV}{dz^2} - \alpha V = 0 \tag{10}$$

of the differential equation (9) is a confluent hypergeometric equation whose solution is known (Abramowitz and Stegun [1]) for integral as well as for non integral values of γ, being given by

$$V = AM(\alpha ; \gamma ; z) + B\,\phi(z) \tag{11}$$

In (11) $M(\alpha ; \gamma ; z)$ is the confluent hypergeometric function such that

$$M(\alpha; \gamma; z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!} \tag{12}$$

Where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ with $(\alpha)_o = 1$.

The series on the right hand side of (12) is known to be convergent (Rainville[6]) for all finite z unless $\gamma = 0$ or a negative integer.

The function $\varphi(z)$ in (11) depends on γ as follows :

$\varphi(z) = z^{1-\gamma}\ M(\alpha+1-\gamma ; 2-\gamma , z)$ when γ is not an integer

$$= \frac{(-1)^{K+1}}{K!\,\Gamma(\alpha-K)} \Big[M(\alpha;\, K+1;\, z)\ln z$$

$$+ \sum_{r=0}^{\infty} \frac{(\alpha)_r z^r}{(k+1)_r\, r!} \left\{\psi(\alpha+r) - \psi(1+r) + \psi(1+K+r)\right\}\Big]$$

$$+ \frac{(K-1)!}{\Gamma(\alpha)} z^{-K} M(\alpha-K;\, 1-K;\, z)_K \tag{13}$$

When γ is an integer and $= K+1$, $(K = 0, 1, 2, \ldots)$, the last function being the sum to K terms. It is to be interpreted as zero when $K = 0$, and $\psi(\alpha) = \dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

The particular solution of (9) is given by

$$V = Z^{\beta}\, [a_0 + a_1 z \;\; {}_2F_2\, (1, \alpha + \beta + 1;\, \beta+2,\, \beta+\gamma+1\, , z)]$$

$= f(z)$, say

Where

$$a_0 = \frac{\eta}{\beta(\beta+\gamma-1)}$$

$$a_1 = \frac{1}{(\beta+1)(\beta+\gamma)} \left[\xi + \frac{\eta(\alpha+\beta)}{\beta(\beta+\gamma+1)}\right]$$

$\beta = p\text{-}n$ and ${}_2F_2\,(a, b; c, d; z)$ is the generalised hypergeometric function such that the series represented by ${}_2F_2\,(a, b; c, d; z)$ is convergent for all finite z, provided, of course, that neither c nor d is zero or a negative integer.

Thus the general solution of (9) is known. Hence using (8) the displacement u is given by

$$u = z^{n}\, [AM\, (\alpha\, ;\, \gamma\, ;\, z) + B\, \phi\, (z) + f\, (z)] \tag{14}$$

Substituting (14) in (2) we get the components of strain. The components of stress may be obtained from (1) together with (3) and (8) as

$$\sigma_r = A\, h_1(z) + B\, h_2(z) + g(z) \tag{15}$$

$$\sigma_\theta = A\, h_3(z) + B\, h_4(z) + f_1(z) \tag{16}$$

$$\sigma_z = A\, h_5(z) + B\, h_6(z) + f_2(z) \tag{17}$$

where

$$f_1(z) = [z^{n-p}\{c_4 f(z) + c_6 z f'(z)\} + e\,\lambda_{23}] \exp(-z),$$

$$f_2(z) = [z^{n-p}\{c_7 f(z) + c_9 z f'(z)\} + e\,\lambda_{33}] \exp(-z),$$

$$g(z) = [z^{n-p}\{c_1 f(z) + c_3 z f'(z)\} + e\,\lambda_{13}] \exp(-z),$$

$$h_1(z) = z^{n-p}[c_1 M(\gamma - \alpha; \gamma; -z) + c_2 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_2(z) = z^{n-p}[c_1 \phi(z) + c_3 z \phi'(z)] \exp(-z),$$

$$h_3(z) = z^{n-p}[c_4 M(\gamma - \alpha; \gamma; -z) + c_5 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_4(z) = z^{n-p}[c_4 \phi(z) + c_6 z \phi'(z)] \exp(-z),$$

$$h_5(z) = z^{n-p}[c_7 M(\gamma - \alpha; \gamma; -z) + c_8 z M(\gamma - \alpha; \gamma + 1; -z)],$$

$$h_6(z) = z^{n-p}[c_7 \phi(z) + c_9 z \phi'(z)] \exp(-z),$$

$$c_1 = k^p (mn\lambda_{11} + \lambda_{12}),$$

$$c_2 = \frac{\alpha}{\gamma}\, m\lambda_{11}\, k^p,$$

$$c_3 = m\lambda_{11}\, k^p,$$

$$c_4 = k^p (mn\lambda_{12} + \lambda_{22}),$$

$$c_5 = \frac{\alpha}{\gamma}\, m\lambda_{12}\, k^p,$$

$$c_6 = m\lambda_{12}\, k^p,$$

$$c_7 = k^p\,(mn\lambda_{13} + \lambda_{23}),$$

$$c_8 = \frac{\alpha}{\gamma}\; m\,\lambda_{13}\; k^p,$$

$$c_9 = m\lambda_{13}\, k^p.$$

Finally the boundary conditions (7) determine A and B as

$$A = [g_2(z_2)\, h_2(z_1) - g_1(z_1) h_2(z_2)]/\Delta$$

$$B = [h_1(z_2)\, g_1(z_1) - h_1(z_1) g_2(z_2)]/\Delta \qquad (18)$$

where

$$\Delta = h_2\,(z_1)\, h_1\,(z_2) - h_1\,(z_1)\, h_2\,(z_2)$$

$$g_1\,(z) = -\,p_0 - g(z),\; g_2\,(\mathrm{z}) = -p_1 - g(z)$$

$$z_1 = ka^m,\; z_2 = kb^m$$

Substitution of (18) in (14), (15), (16) and (17) determines displacement and stresses completely.

A partial check of the results presented in this discussion has been made by making $m \to 0$ and $k \to 0+$. Thus we see from (3) that λ_{ij} are the elastic coefficients in the homogeneous case. Here we shall confine our attention only to the case in which γ is not an integer. Choosing appropriate $\phi(z)$ from (13) and letting m $\to$ 0 and k $\to$ 0 + we get the displacement and stresses as

$$u = \frac{ab}{(s\lambda_{11} + \lambda_{12})}\left[\frac{q_0}{b}\left(\frac{r}{b}\right)^s - \frac{q_1}{a}\left(\frac{r}{a}\right)^s\right] \Big/ \left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]$$

$$+\,\frac{ab}{(\lambda_{12} - s\lambda_{11})}\left[\frac{q_1}{a}\left(\frac{a}{r}\right)^s - \frac{q_0}{b}\left(\frac{b}{r}\right)^s\right] \Big/ \left[\left(\frac{b}{a}\right)^s - \left(\frac{a}{b}\right)^s\right]$$

$$+\ \frac{(\lambda_{23} - \lambda_{13})er}{\lambda_{11} - \lambda_{12}}. \tag{19}$$

$$\sigma_r = \frac{\dfrac{ab}{r}\left\{\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right] + \left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+\ \frac{(\lambda_{23} - \lambda_{13})}{\lambda_{11} - \lambda_{22}}(\lambda_{11} + \lambda_{12})e + \lambda_{13}e \tag{20}$$

$$\sigma_0 = \frac{\dfrac{ab}{r}\left\{\begin{aligned}&\left(\dfrac{s\lambda_{12} + \lambda_{22}}{s\lambda_{11} + \lambda_{12}}\right)\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right]\\ &+\left(\dfrac{\lambda_{22} - s\lambda_{12}}{s\lambda_{12} - \lambda_{11}}\right)\left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\end{aligned}\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+\left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{12} + \lambda_{22})e + \lambda_{23}e \tag{21}$$

$$\sigma_z = \frac{\dfrac{ab}{r}\left\{\begin{aligned}&\left(\dfrac{s\lambda_{13} + \lambda_{23}}{s\lambda_{11} + \lambda_{12}}\right)\left[\dfrac{q_0}{b}\left(\dfrac{r}{b}\right)^s - \dfrac{q_1}{a}\left(\dfrac{r}{a}\right)^s\right]\\ &+\left(\dfrac{\lambda_{23} - s\lambda_{13}}{\lambda_{12} - s\lambda_{11}}\right)\left[\dfrac{q_1}{a}\left(\dfrac{a}{r}\right)^s - \dfrac{q_0}{b}\left(\dfrac{b}{r}\right)^s\right]\end{aligned}\right\}}{\left[\left(\dfrac{b}{a}\right)^s - \left(\dfrac{a}{b}\right)^s\right]}$$

$$+\left(\frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}}\right)(\lambda_{13} + \lambda_{23})e + \lambda_{33}e \tag{22}$$

$$q_0 = p_0 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

$$q_1 = p_1 + \frac{\lambda_{23} - \lambda_{13}}{\lambda_{11} - \lambda_{22}} (\lambda_{11} + \lambda_{12})e + \lambda_{13}e$$

These results are in complete agreement with the results of the homogeneous case.

If the material of the structure be isotropic and homogeneous the material constants λ_{ij} become

$$\lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda + 2\mu \, , \; \lambda_{12} = \lambda_{13} = \lambda_{23} = \lambda$$

In this case $s = 1$ and the displacement and the stress components have the same expressions as those given in Love [5].

Numerical Results

Numerical computations of stresses have been made by taking

$$\lambda_{11} = 40.75 \quad \lambda_{22} = 50.61 \quad \lambda_{33} = 42.63 \quad \lambda_{44} = 15.66$$
$$\lambda_{55} = 19.14 \quad \lambda_{66} = 19.00 \quad \lambda_{23} = 12.76 \quad \lambda_{13} = 12.18$$
$$\lambda_{12} = 18.27,$$

which are the material constants for topaz. Here λ_{ij}'s are expressed in term

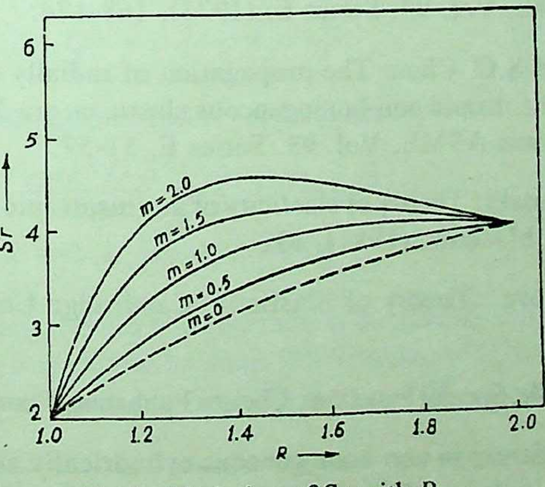

Fig. 1 : Variation of Sr with R

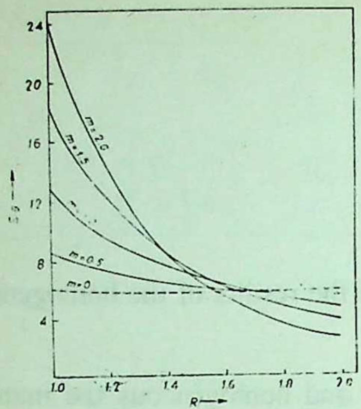

Fig.2 : Variation of S_θ with R

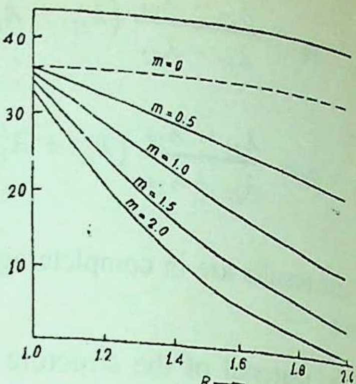

Fig.3 : Variation of S_z with R

of a unit stress of $10^6\ lb/(in)^2$. In addition, we assume that $\delta = 2 \times 10^6\ lb(in)^2$ and write $p_0 = \delta e$, $p_1 = 2\delta e$. With this assumption, we plot the values of $S_r = -\sigma r/e$, $S_\theta = -\sigma \theta/e$ and $S_z = -\sigma z/e$, against $R = (r/a)$ in $1 \le R \le 2$, in Fig. 1, Fig. 2 and Fig.3 respectively. In our calculation we have considered $m = 0.0, 0.5, 1.0, 1.5, 2.0$. From the figures it is observed that the stresses change regularly with variations of m.

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A STUDY OF SHAPE OF ACTIVATED COMPLEX IN REACTION BETWEEN POTASSIUM PEROXODISULPHATE AND POTASSIUM IODIDE

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(Record 17.07.92)

ABSTRACT

The rate constants of the reaction between potassium peroxodisulphate and potassium iodide have been measured in a series of water acetone mixtures at 30± 0.1°C and d40 ± 0.1°C at various ionic strengths (μ). The radii of activated complexes for single sphere and double sphere models have been calculated from the slopes of linear plots of logarithm of rate constant at zero ionic strength (k_0) versus reciprocal of dielectric constant ($1/\varepsilon$). Results show that the description of the activated complex could best be given by the single sphere model.

An attempt has also been made to evaluate the values of electrostatic contributions to the free energies changes ($\Delta G^{\neq}_{e.s}$).

INTRODUCTION

The reaction between peroxodisulphate and iodide ions takes place in two steps [1,2]

$$S_2O_8^{--} + I^- \longrightarrow IS_2O_8^{--} \qquad \text{slow ... (1)}$$

$$IS_2O_8^{---} + I^- \longrightarrow I_2 + 2SO_4^{--} \qquad \text{fast ... (2)}$$

This reaction follows second order kinetics:

$$\frac{-d[S_2O^{--}_8]}{dt} = k[S_2O^{--}_8]\ [I^-]z$$

where 't' is time and 'k' is rate constant.

The influence of dielectric constant on the rate of ionic reactions having similar charges was studied by Amis [3], Scatchard [4], Laidler [5], Laidler-Eyring [6], Christiansen [7], Ghaziuddin - et al. [8-10] and Fahim Uddin et al. [11-13]. Ghaziuddin - et al. [14] and Fahim Uddin et al. [15] also studied the reaction between peroxodisulphate-iodide ions in ethanol water and methanol water mixtures respectively and found that the activated complex is single sphere for both the media i.e. ethanol water and methanol water mixtures.

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In the present work, the reaction between potassium peroxodisulphate and potassium iodide has been studied to confirm whether the activated complex consists of single sphere or double sphere model.

EXPERIMENTAL

Potassium peroxodisulphate, potassium iodide, sodium thiosulphate, potassium dichromate, starch, acetone used were of E. Merck. Double distilled water was used throughout the course of experiments.

Dielectric constants of the medium were varied by changing proportions of water acetone mixtures. The composition of water acetone mixtures *(W/W)* were 8. 08%, 12.25%, 16.51%, 20.07% and 25.32% and respective values of dielectric constants were taken from Akerlof [16-17].

The rate constants of the reaction were measured by a titrimetric method. Calculated volumes of $K_2S_2O_8$ and KI were taken in 25 ml volumetric flasks and diluted up to the mark. These two diluted solutions were kept in a thermostatic bath (Type K-33 Haake, Karlsruhe, Germany) at constant temperature for half an hour. Then these two solutions were mixed and the time was recorded. A 5 ml portion of this reaction mixture was withdrawn after each interval of 5 minutes. The mixture was chilled to freeze the reaction and titrated against a standard solution of sodium thiosulphate. 2 drops of starch solution was added as an indicator.

RESULTS AND DISCUSSIONS

The rate constants were calculated using the integrated form of the rate expression for a second order reaction with the same initial concentrations i.e.,

$$k = \frac{1}{t} \cdot \frac{x}{a(a-x)} \qquad \ldots\ldots\ldots(4)$$

where *'a'* is the initial concentration of potassium peroxodisulphate and potassium iodide, x is the concentration of the product at time *'t'* and *(a-x)* is the remaining concentration of the reactant at time *'t'*.

Plots of *[x/a(a-x)]* against time 't' were drawn at different ionic strengths and composition of solute i.e. acetone water mixtures. A representative plot of *[x/a(a-x)]* versus '*t*' at ionic strength (μ) = 7.84 x 10^{-2} mil.dm^{-3}in 10% acetone water mixtures at 40°C is shown in figure - 1. From the slopes of the plots, rate constants were evaluated. The values of rate constants for the

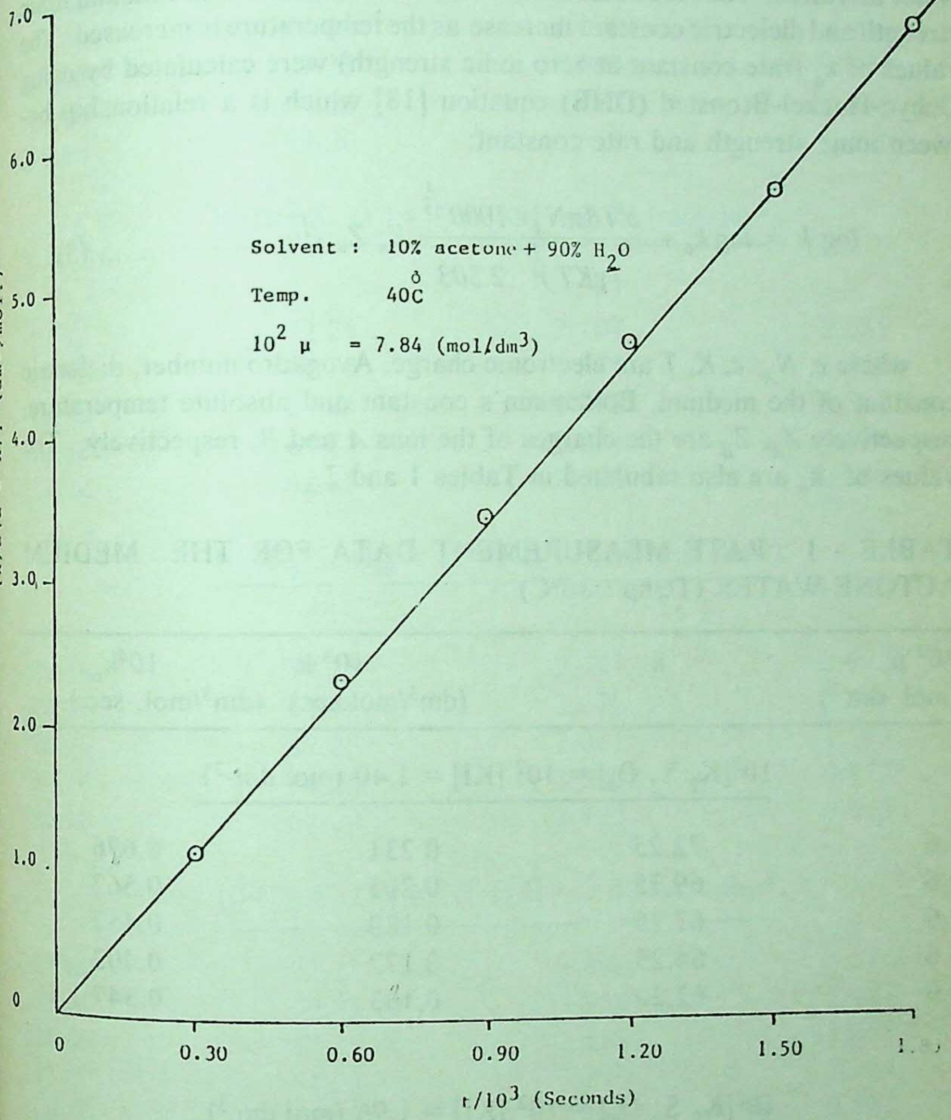

FIGURE - 1

Fig 1 : Plot of *[x/a (a-x)]* versus '*t*' at 40^0C

reaction between peroxo-disulphate and iodide ions in acetone water mixtures at 30° and 40°C are summarized in Tables 1 and 2 respectively.

It was also noted that for the same value of dielectric constant, the rate constant increased with increasing ionic strength. These results are in agreement with those reported earlier [14] for the same reaction in ethanol water mixtures. The results also show that the rate constant at constant ionic strength and dielectric constant increase as the temperature is increased. The values of k_o (rate constant at zero ionic strength) were calculated by using Debye-Huckel-Bronsted (DHB) equation [18] which is a relationship between ionic strength and rate constant:

$$\log k = \log k_0 + \frac{e^3 (8\pi N_A / 1000)^{\frac{1}{2}}}{(\varepsilon KT)^{\frac{3}{2}} \, .2.303} . Z_A . Z_B . \sqrt{\mu} \qquad \text{........(5)}$$

where e, N_A, ε, K, T are electronic charge, Avogadro number, dielectric constant of the medium, Boltzmann's constant and absolute temperature, respectively Z_A, Z_B are the charges of the ions A and B, respectively. The values of k_o are also tabulated in Tables 1 and 2.

TABLE - 1 : RATE MEASUREMENT DATA FOR THE MEDIUM ACTONE-WATER (Temp : 30°C)

$10^2\ \mu$ (mol. dm^{-3})	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol. sec.)
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] $= 1.40$ (mol.dm^{-3})		
5.6	72.25	0.231	0.676
5.6	69.75	0.205	0.562
5.6	67.25	0.180	0.457
5.6	64.25	0.172	0.403
5.6	62.20	0.163	0.347
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] $= 1.96$ (mol.dm^{-3})		
7.84	72.25	0.677	1.585
7.84	69.75	0.625	1.349
7.84	67.25	0.419	0.832

7.84	64.25	0.342	0.616
7.84	62.20	0.278	0.447
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 2.52$ (mol.dm^{-3})			
10.08	72.25	1.820	3.467
10.08	69.75	1.090	1.905
10.08	67.25	0.963	1.549
10.08	64.25	0.872	1.259
10.08	62.20	0.653	0.830
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 3.08$ (mol.dm^{-3})			
12.32	72.25	2.167	2.089
12.32	69.75	1.318	1.950
12.32	67.25	1.042	1.380
12.32	64.25	0.983	1.148
12.32	62.20	0.876	0.849
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 5.04$ (mol.dm^{-3})			
20.16	72.25	2.777	2.692
20.16	69.75	2.670	2.291
20.16	67.25	2.031	1.513
20.16	64.25	1.538	0.977
20.16	62.20	1.495	0.813
$10^2[K_2\ S_2\ O_8] = 10^2\ [KI] = 7.00$ (mol.dm^{-3})			
28.00	72.25	5.483	3.516
28.00	69.75	3.958	2.188
28.00	67.25	3.513	1.659
28.00	64.25	3.230	1.267
28.00	62.20	2.602	0.832

TABLE - 2 RATE MEASUREMENT DATA FOR THE MEDIUM ACETONE WATER (Temp : 400C)

10^2 μ (mol. dm-3) sec.)	ε	10^3 k (dm^3/mol.sec)	$10^4 k_o$ (dm^3/mol.
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 1.40 (mol.dm-^{3})		
5.60	69.00	1.967	5.623
5.60	66.75	1.823	4.898
5.60	64.25	1.592	3.981
5.60	62.00	1.102	2.521
5.60	59.75	1.061	2.239
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 1.96 (mol.dm^3)		
7.84	69.00	3.816	8.709
7.84	66.75	3.683	7.762
7.84	64.25	2.516	4.786
7.84	62.00	1.616	2.818
7.84	59.75	1.482	2.344
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 2.52 (mol./dm^3)		
10.08	69.00	4.762	8.851
10.08	66.75	4.333	7.431
10.08	64.25	3.333	5.129
10.08	62.00	3.000	4.169
10.08	59.75	3.563	3.162
	$10^2[K_2 S_2 O_8] = 10^2$ [KI] = 3.08 (mol./dm^3)		
12.32	69.00	6.066	9.332
12.32	66.75	5.867	8.318
12.32	64.25	3.533	4.467
12.32	62.00	3.492	3.935
12.32	59.75	3.432	3.388

	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 5.04$ (mol./dm^3)		
20.16	69.00	8.108	7.499
20.16	66.75	6.900	5.623
20.16	64.25	5.283	3.715
20.16	62.00	4.398	2.692
20.16	59.75	4.385	2.399
	$10^2[K_2 S_2 O_8] = 10^2 [KI] = 7.00$ (mol./dm^3)		
28.00	69.00	8.903	5.370
28.00	66.75	7.433	3.890
28.00	64.25	6.583	2.884
28.00	62.00	5.000	1.738
28.00	59.75	4.842	1.493

The effect of dielectric constant on rate constant can be explained by the Laidler-Eyring equation [6] for single and double sphere models respectively. The equations are expressed as followed.

$$\ln k_0 = \ln k_\infty - \frac{e^2}{2\varepsilon KT}\left[\frac{Z^2{}_A}{r^{\neq}} - \frac{Z^2{}_A}{r_A} - \frac{Z^2{}_B}{r_B}\right] \quad \text{(for single sphere mode) ...(6)}$$

$$\ln k_0 = \ln k_\infty - \frac{Z_A . Z_B . e^2}{KTr_{AB}} . \frac{1}{\varepsilon} \quad \text{(for double sphere model)...(7)}$$

where k_∞ is the rate constant at zero ionic strength and infinite dielectric constant. $r^{\neq}$ and r_{AB} are the radii of the activated complex for the single model and the double sphere model, respectively, and r_A, r_B are radii of the ions A and B, respectively. The plots of logarithm of rate constant at zero ionic strength ($\log k_o$) against the reciprocal of the dielectric constant of the medium $(1/\varepsilon)$ at various ionic strengths and temperatures were straight lines with negative slopes. A representative graph is shown in Figure - 2.

The experimental values of $r^{\neq}$ and r_{AB} calculated from the slopes of straight lines using equations (6) and (7) are tabulated in Table - 3.

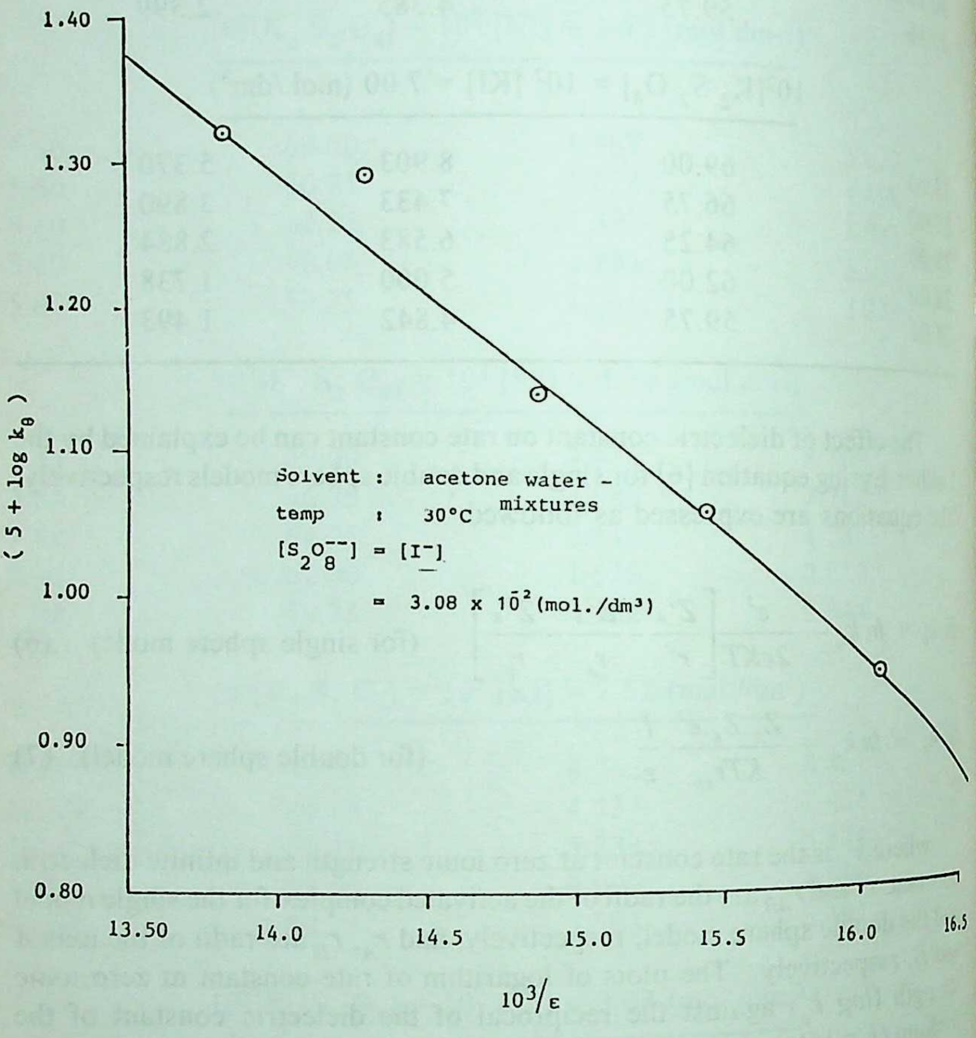

FIGURE - 2

Fig. 2 : Plot of log k_o versus '$1/\varepsilon$' in aqueous acetone medium at 30°C.

TABLE - 3 : CALCULATED VALUES OF RADII OF ACTIVATED COMPLEX

$10^2[K_2S_2O_8]$ (mol/dm^3)	10^2[KI] (mol/dm^3)	$r^{\neq}$ (Å)	r_{AB} (Å)
	Temp : 30°C		
1.40	1.40	3.04	3.74
1.96	1.96	2.23	1.86
2.52	2.52	2.25	1.89
3.08	3.08	2.70	2.77
5.05	5.04	2.26	1.91
7.00	7.00	2.19	1.81
		Average	Average
		2.44 ± 0.37	2.33 ± 0.83
	Temp : 40°C		
1.40	1.40	2.57	2.48
1.96	1.96	2.13	1.72
2.52	2.52	2.49	2.33
3.08	3.08	2.44	2.22
5.05	5.04	2.30	1.98
7.00	7.00	2.20	1.83
		Average	Average
		2.35 ± 0.2	2.09 ± 0.32

The radius of the peroxodisulphate ion was estimated on the assumption that the volume of the molecule of a compound which is supposed to be spherical is a sum of volumes of individual ions consituting the compound as follows :

$$V_{S_2O_8^{2-}} = 2V_{S6} + + 8V_{0^{2-}}$$

where V is the volume of the ion.

$$\frac{4}{3}\pi r^3_{S_2O_8^{2-}} = 2(\frac{4}{3}\pi r_S^3 6 + \; + \; 8(\frac{4}{3}\pi r^3_{O^{2-}})$$

$$r_{S_2O_8^{2-}} = 2.80 \text{ Å}$$

$r_{I^-} = 2.16$ Å (taken from Emeleus [19] ....(11)

The radius of activated complex for single sphere model ($r^{\neq}$) is calculated on the basis of assumption that the volume of single sphere activated complex ($V^{\neq}$) is the sum of volume of reactants.

$$V^{\neq} = V_{I^-} + V_{S_2O_8^{2-}} \qquad \text{........(12)}$$

$$r^{\neq} = 3.176 \text{ Å} \qquad \text{........(13)}$$

The radius of activated complex for double sphere model is calculated assuming that the radius of activated complex is the sum of the radii of reactants.

$$r_{AB} = r_{.S_2O_8^{2-}} + r_{I^-} \qquad \text{........(14)}$$

$$= 2.80 + 2.16 = 4.96 \text{ Å} \qquad \text{........(15)}$$

In all calculations the values of radii of individual ions were taken from Emeleus [19] and confirmed by Wilson [20]. It has been concluded from the comparison of average experimental values of $r^{\#}$ and r_{AB} with their theoretical values that the shape of the activated complex could best be given by single sphere model for reaction between peroxodisulphate and iodide ions because the value of $r^{\#}_{expt}$ is more closer to the theoretical value of single sphere model. The results show that change in temperature has no effect on the shape of activated complex. The values of $r^{\#}$ and r_{AB} are slightly less than the values given by Ghaziuddin *et al*. [14]. This may be due to the volatile nature of acetone.

Electrostatic contributions to the changes in free energy of activation $\Delta G^{\#}_{es}$. in the formation of activated complex for single sphere model were evaluated using the following expression [5] :

$$\Delta G^{\neq}_{e.s} = \frac{N_A e^2}{2}\left[\frac{(Z_A + Z_B)^2}{r^{\neq}} - \frac{Z^2{}_A}{r_A} - \frac{Z^2{}_B}{r_B}\right] \qquad ....(16)$$

The values of $\Delta G^{\neq}_{es}$ at different temperatures i.e. 30^0 and $40\,^0$C and ionic strengths as a function of dielectric constant are summarised in Table - 4.

ABLE - 4 : ELECTROSTATIC CONTRIBUTION TO $\Delta G_{es}^{\neq}$

$10^2\ \mu$ (mol/dm^3)	Dielectric constant (ε)				
	72.25	69.75	67.25	64.75	62.20
			Temp : 30°C		
5.60	1.028	1.065	1.105	1.147	1.194
7.84	2.062	2.136	2.215	2.300	2.394
10.08	2.027	2.100	2.178	2.262	2.235
12.32	1.386	1.436	1.489	1.547	1.610
20.16	2.010	2.083	2.160	2.243	2.335
28.00	2.133	2.209	2.291	2.380	2.477
	Dielectric constant (ε)				
	69.00	66.75	64.25	62.00	59.75

Temp : 40°C $_{e.s}$(KJ/mole)

5.60	1.621	1.676	1.741	1.804	1.872
7.84	2.349	2.429	2.523	2.615	2.713
10.08	1.735	1.793	1.863	1.930	2.003
12.32	1.809	1.870	1.943	2.014	2.089
20.16	2.035	2.103	2.186	2.265	2.350
28.00	2.214	2.289	2.378	2.464	2.555

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ON A LAPLACE - HARDY TRANSFORMATION

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ABSTRACT

In this paper we shell give a few fundamental theorems that depict certain properties of Laplace-Hardy transformation that seem to be interesting and significant.

INTRODUCTION

In the present paper for the Classical Laplace-Hardy transformation, we shall use the notation:

$$(1).\ \phi[f : p.q] = pq\int_0^\infty\int_0^\infty e^{-px-qy} C_\nu(pxqy)\, f(px, qy)\,dx\,dy$$

where $f(px, qy)$ is suitably chosen function under a certain set of condition on ν and α given by Pathak and Pandey, [5].
and

$$(2).\ C_\nu(z) = \cos(a\pi) J_\nu(z) + \sin(a\pi)\, Y_\nu(z)$$

$J_\nu(z)$ and $Y_\nu(Z)$ being Bessel functions [(2), p.38] of the first and second kind respectively, and (2) valid under certain conditions given by Erdelyi [2].

Throughout this work we use the notation:

$$(3).\ \phi[f : p.q] \overset{C_\nu}{=} f(px, qy).$$

The C_ν-function is defined by (2), and if we take $a = 0$, we obtain $C_\nu(z) = J_\nu(z)$ And If $a = 1/2$, we obtain $C_\nu(z) = y_\nu(z)$. And if $\nu = 1/2$, we obtain $C_\nu(z) = [(1/2)\pi z]^{-1/2} \sin(z - a\pi)$. So by using the conditions on 'ν' and 'a' we obtain some special cases of Hardy's transform.

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Theorem

If $\phi_i\,[f_i : p,q\,] \overset{C_\nu}{\underset{\cdot\cdot}{=}} f_i\,(px,\,qy)$, $i = 1,2.$

then

$$(4).\quad \int_0^\infty\int_0^\infty \phi_1[f_1 : p,q]\, f_2(px,qy)\,\frac{dp}{p}\,\frac{dp}{q}$$

$$= \int_0^\infty\int_0^\infty f_1(px,qy)\;\phi_2[f_2 : p,q]\,\frac{dp}{p}\,\frac{dp}{q}$$

provided that the integral involved are absolutely convergent.

Proof

By putting the value of $\phi_1\,[\,f_1 : p,q\,]$ on the left hand side of (4), we have

$$\int_0^\infty\int_0^\infty \{\, pq\int_0^\infty\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_1(px,qy)\,dxdy\,\}\,f_2(px,qy)\,\frac{dp}{p}\,\frac{dp}{q}$$

Now changing the order of integration, we get

$$= \int_0^\infty\int_0^\infty \{\, pq\int_0^\infty\int_0^\infty e^{-pxqy}C_\nu(px,qy)\,f_2(px,qy)\,\frac{dp}{p}\,\frac{dp}{q}\,\}\, f_1(px,qy)\,dxdy.$$

$$= \int_0^\infty\int_0^\infty f_1(px,qy)\{\int_0^\infty\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_2(px,qy)\,dp\,dq\,\}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty f_1(px,qy)\{\,xy\int_0^\infty\int_0^\infty e^{-pxqy}C_\nu(pxqy)\,f_2(px,qy)\,dp\,dq\,\}\,\frac{dx}{x}\,\frac{dy}{y}$$

$$= \int_0^\infty\int_0^\infty f_1(px,qy)\;\phi_2[f_2 : p,q]\,\frac{dx}{x}\,\frac{dy}{y}$$

$$= \int_0^\infty\int_0^\infty f_1(px,qy)\;\phi_2[f_2 : p,q]\,\frac{dp}{p}\,\frac{dq}{q}$$

Hence the theorem.

Theorem 2

Let $\phi\,[f : p,q] \overset{C_\nu}{\underset{\cdot\cdot}{=}} f(px,qy) = g(p,q)$

and $\phi\,[h : p, q] \overset{C_\upsilon}{\underset{\cdot\,\cdot}{=}} h\,(px.qy) = \psi\,(p.q)$

then

$$(5).\quad \int_0^\infty\int_0^\infty \psi(u,v)\, f(ux,vy)\,(uv)^{-1}du\,dv$$

$$= \int_0^\infty\int_0^\infty g(u,v)\, h(ux,vy)\,(uv)^{-1}du\,dv$$

provided that either of the integral in (5) is absolutely convergent and the ϕ-transformation of $|f(ux, vy)|$ and $|h\,(ux.vy)|$ exist.

Proof

In view of theorem 1:

$$(6).\ \psi(u.v) = uv\int_0^\infty\int_0^\infty e^{-uxvy}C_\upsilon(uxvy)\, h(ux.vy)\,dx\,dy$$

Now putting the value $\psi\,(u.v)$ on the left hand side of (5) :

$$\int_0^\infty\int_0^\infty \{uv\int_0^\infty\int_0^\infty e^{-uxvy}C_\upsilon(uxvy)\, h(ux.vy)\,dx\,dy\}\, f(ux.vy)(uv)^{-1}\,du\,dv$$

changing the order of integration:

$$= \int_0^\infty\int_0^\infty \{uv\int_0^\infty\int_0^\infty e^{-uxvy}C_\upsilon(uxvy)\, f(ux.vy)\,(uv)^{-1}du\,dv\}\, h(ux.vy)\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \{xy\int_0^\infty\int_0^\infty e^{-uxvy}C_\upsilon(uxvy)\, f(ux.vy)\,du\,dv\}\, h(ux.vy)\,(xy)^{-1}dx\,dy$$

$$= \int_0^\infty\int_0^\infty g(x,y)\, h(ux,vy)\,(xy)^{-1}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty g(u,v)\, h(ux,vy)\,(uv)^{-1}\,du\,dv.$$

Hence the theorem.

Presently numerical inversion of certain integral transforms are being studied in order to make them computer oriented. [6].

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ON A NONLINEAR FUNCTIONAL INTEGRODIFFERENTIAL SYSTEM

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ABSTRACT

The aim of the present paper is to study the existence, uniqueness and other properties of the solutions of a nonlinear functional integrodifferential system of more general type by using monotone iterative method, comparison principle, the notion of upper and lower solutions and integral inequality established by Pachapatte.

Mathematical Subject Classification: 47HO5 47H10

Keywords and Phrases: Integrodifferential System, Monotone iterative technique.

INTRODUCTION

In what follows, we denote by R^n the *n-dimensional Euclidean space* with the Euclidean norm $\|\cdot\|$ .Let $I_o = [-r, 0]$, $I = [t_0, t_o+T]$, $J = [t_0 - r, t_0+T]$. $r > 0$, $t_o \geq 0$ and $T > o$ ar econstants. Let $C = C\,[I_0, R^n]$ be the Banach space of continuous functions defined on I_o with supremum norm $\|\cdot\|_c$. We use the notation $< v, w >$ to denote the segment in $C\,[J, R^n]$ defined by $<v, w> = \{x \in C\,[J, R^n] : v(t) \leq x(t) \leq w(t),\ t \in J\}$. If x is a continous function from J to R^n and $t \in I$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ *for* $\theta \varepsilon I_\sigma$. In the present paper, we investigate the existence, uniqueness and other properties of the solution of following nonlinear functional intergro differential system

$$x'(t) = f\left(t, x(t), x_t, \int_{t_0}^{t} k\left(t, s, x(s), x_s\right)ds\right), \quad t \in I.$$

(1.1)

$$x_{t_0}(\theta) = \phi(\theta), \quad \theta \in I_0$$

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where $x : J \to R^n$, $K \in C[I \times I \times R^n \times C, R^n]$, $f \in C[I \times R^n \times C \times R^n, R^n]$ and $0 \in C[I_0, R^n]$ is a given function.

The problems of existence, uniqueness and other properties of solutions of special forms of the system (1.1) have been studied by many authors in the literature by using different techniques, for example, see [2,4,6,7,9,10,12] and the references given therein. In [9], V. Lakshmikantham and B.G. Zhang have proved the existence of extremal solution of the equation (1.1) when k = 0 by using monotone iterative technique. B.G. Pachpatte [12] has established the existence of extermal solutions of the system (1.1) without functional argument. The main tools employed in our analysis are based on the monotone iterative method, comparision principle, the notion of upper and lower solutions and integral inequality established by Pachpatte.

We organize the paper as follows: In section 2, we present the preliminaries and statement of our main results, the proofs of which are contained in sections 3 and 4.

PRELIMINARIES AND STATEMENT OF RESULTS

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion. Throughout this paper, without further mention, we assume that all inequalities between vectors are componentwise

Definition 1. The function $w \in C[J, R^n] \cap C'[I, R^n]$ is called an *upper solution* of (1.1.) if

$$w'(t) \geq f\left(t, w(t), w_t, \int_{t_0}^{t} k\left(t, s, w(s), w_s\right) ds\right), \quad t \varepsilon I$$

$$w_{t_0}(\theta) \geq \phi(\theta), \quad \theta \in I_0$$

Similarly, a function $v \in C[J, R^n] \cap C'[I, R^n]$ is called a *lower solution of* (1.1) if

$$v'(t) \leq f\left(t, v(t), v_t, \int_{t_0}^{t} k\left(t, s, v(s), v_s\right) ds\right), \quad t \in I.$$

$$v_{t_0}(p) \leq \phi(\theta). \quad \theta \in I_0$$

Definition 2. The functions α and β, $\alpha, \beta \in C\,[J, R^n] \cap C'\,[I, R^n]$ are called minimal and maximal solutions of (1.1) respectively, if every other solution $x \in C\,[J, R^n] \cap C'\,[I, R^n]$ of (1.1) satisfies the relation

$$\alpha(t) \leq x(t) \leq \beta(t), \; t \varepsilon J.$$

Definition 3. The solution $x(t)$ of (1.1) is said to be uniformly slowly growing if for every $\gamma > 0$ there exists a constant Q, possibly depending on γ, such that the indequality

$$\|x_t\|_C \leq Q\,\|\phi\|_C \cdot e^{\gamma(t-t_0)}, \; t \geq t_0$$

holds for $\|\phi\|_C < \infty$

we require the following lemmas in our further discussion.

Lemma 1. (See, [9,p.228]). Assume that $u \in C\,[J, R] \cap C'\,[I, R]$ and

$$u'(t) \leq -Mu(t) - N \int_{-r}^{0} u_t(s)\,ds,$$

$t \in I$. Further assume that

i) either $u(t_0) \leq u_{t_0}(s) \leq 0$, $s \in I_0$ and $(M + Nr)\,T \leq 1$
   or
ii) $u_{t_0}(s) \leq 0$, $s \in I_0$, $u \in C'([t_0 - r, t_0], R)$ and

$$u'(t) \leq \frac{\lambda}{T + r}, \; t \varepsilon [t_0 - r, t_0]$$

where $\min\limits_{[t_0 - r; t_0]} u(t) = -\lambda$, $\lambda \geq 0$ and $(M + Nr)\,(T + r) \leq 1$. Then $u(t) \leq 0$ on I.

Lemma 2. (See [11, p.758)]. *Let $u(t)$, $m(t)$ and $n(t)$ be real* valued nonnegative continuous functions defined on R^+, for which the inequality

$$u(t) \leq u_0 + \int_0^t m(s)\,u(s)\,ds + \int_0^t m(s)\left[\int_0^s n(\tau)\,u(\tau)\,d\tau\right]ds.$$

holds for all $t \in R^+$, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0\left[1 + \int_0^t m(s)\exp\left\{\int_0^s \left(m(\tau) + n(\tau)\right)d\tau\right\}ds\right].$$

for all $t \in R^+$

For convenience, we list the following hypotheses.

(H_1) The functions $v, w \in C, [J, R^n] \cap C' [I, R^n]$ with $v(t) \leq w(t)$ on J are lower and upper solutions of (1.1).

(H_2) For each $i, 1 \leq i \leq n$, $k_i\,(t, s, x, \phi)$ is *monotone nondecreasing* in x and ϕ for fixed t and s, and $f_i(t, x, \phi, u)$ is *monotone nondecreasing* in u for fixed t, x and ϕ.

(H_3) For each $i, 1 \leq i \leq n$, there exist constants $M, N, \geq 0$ such that

$$f_i\,(t, x, \phi, u) - f_i\,(t, \bar{x}, \bar{\phi} u)$$

$$\geq - M\left(x_i - \bar{x}_i\right) - N \int^{0} \left[\phi_i(s) - \bar{\phi}_i(s)\right] ds$$

whenever $v(t) \leq \bar{x} \leq x \leq w(t)$ and $v_t \leq \bar{\phi} \leq \Phi \leq w_t$ for $t \in I$.

(H_4) The difference $v_{t_0} - \phi$, $\phi - w_{t_0}$ satisfy either the assumption (i) or (ii) of lemma [1]

(H_5) There exist nonnegative constants M_1, M_2, M_3, M_4 and M_5 such that

(2.1) $\| K(t,s,x,\phi) - k(t,s,\bar{x}\,\bar{\phi}) \| \le M_1 // x - \bar{x} \| + M_2 \| \phi - \bar{\phi} \|_c$

(2.2) $\| f(t,x,\phi,u) - f(t,\bar{x}\,\bar{\phi},\ \bar{u}) \|$

$$\le M_3 \| x - \bar{x} \| + M_4 \| \phi - \bar{\phi} \|_C + M_4 \| u - \bar{u} \|$$

(H_6) For $t,\ s \in [t_0,\ \infty),\ \phi \in C,\ x,\ u \in R^n$,

(2.3) $\| K(t,s,x,\phi) \| \le L_i(s)\ [\ \| x \| + \| \phi \|_C\ ]$

(2.4) $\| f(t,x,\phi,u) \| \le L_2(t)\ [\ \| x \| + \| \phi \|_C + \| u \|\ ]$

where $L_1,\ L_2 \in C([\ t_0\ \infty),\ [\ t_0\ \infty))$
and

(2.5) $\int_{t_0}^{\infty} L_1(s)ds < \infty,\ \int_{t_0}^{\infty} L_2(s)ds < \infty.$

(H_7) For $t,\ s \in [t_0,\ \infty],\ \gamma > 0,\ \phi \in C,\ x,\ u \in R^n$,

(2.6) $\| K(t,s,x,\phi) \| \le e^{-\gamma(s-t)} L_1(s)\ [\ \| x \| + \| \phi \|_C\ ]$

(2.7) $\| f(t,x,\phi,u,) \| \le e^{-\gamma t} L_2(t)\ [\ \| x \| + \| \phi \|_C + \| u \|\ ]$

where $L_1,\ L_2 \in C([t_0,\ \infty\ ),\ [t_0,\ \infty\ ))$ and (2.5) holds.

Our main results are established in the following theorems.

THEOREM 1. Let the hypotheses (H_1) - (H_4) hold. Then there exists monotone nondecreasing sequence $\{v_n\}$ and monotone non-increasing sequence $\{w_n\}$ such that $v_n \to \alpha,\ w_n \to \beta$ as $n \to \infty$ uniformly on I and that α, β are minimal and maximal solutions of the system (1.1) respectively. That is , if x is any solution of (1.1)in $<v,w>$, then

$$v \le v_1 \le \ldots \le v_n \le \ldots \le \alpha \le x \le \beta \le \ldots \le w_n \le \ldots \le w_1 \le w$$

THEOREM 2. Let the hypotheses (H_1) - (H_5) hold. Then the maximal solution $\beta(t)$ and the minimal solution $\alpha(t)$ obtained in Theorem 1 coincide on J, that is $\beta(t) = \alpha(t)$, for $t \in J$.

In the following theorems, we assume that solutions of the system (1.1) exist on $[t_o - r, \infty)$.

THEOREM 3. Let the hypothesis (H_6) holds. Then any solution $x(t)$ of the system (1.1) is bounded on $[t_o - r, \infty)$.

THEOREM 4 Let the hypothesis (H_7) holds. Then any solution $x(t)$ of the system (1.1) is uniformly slowly growing.

Remark 1. We note that in [8], G.S. Ladde and B.G. Pachpatte, have established existence of the solution of the equation of type (1.1) by using different conditions and techniques. It is also important to note that M.B. Dhakne and B.G. Pachpatte in [5] have investigated the problems of boundedness, stability, asympotic behaviour and other properties of the solutions of an abstract functional integro-differential equation by using the semigroups method.

PROOFS OF THEOREMS 1 AND 2

For any $\eta \, \varepsilon \, C \, [J, R^n]$ such that $v \leq \eta \leq w$ on J, we consider the following linear system

$$x'(t) = f\left(t, \eta(t), \eta_t, \int_{t_0}^{t} k(t, s, \eta(s), \eta_s) ds\right) - M[x(t) - \eta(t)]$$

$$(3.1) \qquad - N \int_{-r}^{0} [x_t(s) - \eta_t(s)] ds, \; t \in I.$$

$$x_{t_0}(\theta) = \phi(\theta), \; \theta \in I_0$$

Clearly, the linear system (3.1) possesses a unique solution $x(t)$ on J, for each η. We, now, define a mapping A by $A\eta = x$.

We prepare a lemma which forms the basis of our main result.

Lemma 3. Let the hypotheses (H_1) - (H_4) hold. Then

(i) $v \leq Av$, $w \geq Aw$ on J;

(ii) A is a monotone operator on the segment $<v,w>$

Proof : Set $Aw = x$ where x is the unique solution of (3.1) corresponding to w, and $z_i(t) = x_i(t) - w_i(t)$ for each i, $1 < i < n$. Then, we have

$$z'_i(t) = x'_i(t) - w'_i(t)$$

$$\leq f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s),w_s)ds\right) - M[x_i(t) - w_i(t)]$$

$$-N\int_{-r}^{0}\left[x_{i_t}(s) - w_{i_t}(s)\right]ds \; - f_i\left(t, w(t), w_t, \int_{t_0}^{t} k(t,s,w(s),w_s)ds\right)$$

$$= -Mz_i(t) - N\int_{-r}^{0} z_{i_t}(s)\,ds$$

and $z_{i_{t_0}} = \phi_i - w_{i_{t_0}} \leq 0$. By an application of lemma1, $z_i(t) = x_i(t) - w_i(t) \leq 0$ on I, that is, for each i, $1 \leq i \leq n$, $w_i(t) \geq x_i(t)$ for $t \in J$. Hence, $w(t) \geq x(t) = Aw(t)$ on J. Similarly, we show that $v \leq Av$ on J. This proves(i).

Let $p, q \in < v, w >$ and $p \leq q$. Suppose that $Ap = x$ and $Aq = y$. For each i, $1 \leq i \leq n$, set $z_i(t) = x_i(t) - y_i(t)$. Then

$$z_i'(t) = x_i'(t) - y'_i(t)$$

$$= f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, p(s), p_s)\,ds) - M[x_i(t) - p_i(t)]$$

$$-N\int_{-r}^{0} [x_{i_t}(s) - p_{i_t}(s)]\,ds - f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds)$$

$$+ M[y_i(t) - q_i(t)] + N\int_{-r}^{0}\left[y_{i_t}(s) - q_{i_t}(s)\right]ds$$

$$\leq f_i(t, p(t), p_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds$$

$$- f_i(t, q(t), q_t, \int_{t_0}^{t} k(t, s, q(s), q_s)\,ds) - M[x_i(t) - p_i(t)]$$

$$- N \int_{-r}^{0} [\, x_{i_t}(s) - p_{i_t}(s)]\, ds \;+ M\, [y_i(t) - q_i(t)]$$

$$+ N \int_{-r}^{0} [\, y_{i_t}(s) - q_{i_t}(s)]\, ds$$

$$= - Mz_i(t) - N \int_{-r}^{0} z_{i_t}(s)\, ds$$

and $z_{i_{t_0}} = \phi_i - \phi_i = 0$. By lemma *1*, $z_i(t) = x_i(t) - y_i(t) \le 0$ on I, that is for each i, $1 \le i \le n$, $x_i(t) \le y_i(t)$ for $t \in J$. Hence, $x(t) \le y(t)$ on J. Thus, A is a monotone operator on $<v,w>$. This completes the proof of lemma 3.

In order to prove Theorem 1, define the sequences $\{ v_n \}$ and $\{ w_n \}$ as follows :

$$v_1 = Av,\ v_n = Av_{n-1}\,,\ n = 2,3,\ldots$$

$$w_1 = Aw,\ w_n = Aw_{n-1}\,,\ n = 2,3,\ldots$$

on J. by an application of lemma 3, we get

$$v \le v_1 \le v_2 \le \ldots \le v_n \le w_n \ldots \le w_2 \le w_1 \le w$$

on J. Using standard arguments (see [1], [3]), it is easy to show that

$$\lim_{n\to\infty} v_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} w_n = \beta$$

uniformly on J and that α, β are solutions of the system (1.1) on J.

Let $x(t)$ be any solution of the system (1.1) such that $v(t) \le x(t) \le w(t)$ on J. For each i, $1 \le i \le n$, set $z_i(t) = v_{1_i}(t) - x_i(t)$. Then

$$z'_i(t) = v_{1_i}{}'(t) - x'_i(t)$$

$$= f_i(t,\ v(t),\ v_t,\ \int_{t_0}^{t} k(t,\ s,\ v(s),\ v_s)\, ds) - M\, [v_{1_i}(t) - v_i(t)]$$

$$-N\int_{-r}^{0}[v_{1_{i_t}}(s) - v_{i_t}(s)]\,ds - f_i\Big(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\,ds\Big)$$

$$\le f_i\Big(t, v(t), v_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\,ds\Big),$$

$$- f_i\Big(t, x(t), x_t, \int_{t_0}^{t} k(t, s, x(s), x_s)\,ds\Big)$$

$$- M[v_{1_i}(t) - v_i(t)] - N\int_{-r}^{0}[v_{1_{i_t}}(s) - v_{i_t}(s)]\,ds$$

$$\le M[x_i(t) - v_i(t)] + N\int_{-r}^{0}[x_{i_t}(s) - v_{i_t}(s)]\,ds$$

$$- M[v_{1_i}(t) - v_i(t)] - N\int_{-r}^{0}[v_{1_{i_t}}(s) - v_{1_i}(s)]\,ds$$

$$= -Mz_i(t) - N\int_{-r}^{0} z_{i_t}(s)\,ds$$

and $z_{i_{t0}} = v_{1_{t0}} - x_{i_{t0}} = \phi_i - \phi_i = 0$ Then by an application of Lemma 1, we obtain $z_i(t)\ v_{1_i}\ 0$ on J.That is : for each i, $1 \le i \le n$, $v_{1_i}(t) \le x_i(t)$, $t \in J$. Hence $v_1(t) \le x(t)$ for $t \in J$. By induction, it can be shown that $v_n \le x$ for every $n \ge 2, 3, \ldots$ .Similarly, $x \le w_n, n = 1, 2, \ldots$ .Therefore, $v_n \le x \le w_n$ and hence $\alpha \le x(t) \le \beta$ on J.That is : α, β are minimal and maximum solutions of the system (1.1) respectively. This completes the proof of the Theorem 1.

The function $\beta(t)$ and $\alpha(t)$ are solutions of the system (1.1). so we have,

$$\beta(t) = \phi(0) + \int_{t_0}^{t} f\Big(s, \beta(s), \beta_s, \int_{t_0}^{s} k(s, \tau, \beta(\tau), \beta_\tau)\,d\tau\Big)ds,\ t \in I,$$

(3.2)

$$\beta(t) = \phi(t-t_0),\ t_0 - r \le t \le t_0$$

and

$$\alpha(t) = \phi(0) + \int_{t_0}^{t} f(s, \alpha(s), \alpha_s, \int_{t_0}^{s} k(s, \tau, \alpha(\tau), \alpha_\tau)\, d\tau) ds, \; t \in I,$$

(3.3)

$$\alpha(t) = \phi(t-t_0), \; t_0 - r \le t \le t_0$$

Define $z(t) = \|\beta(t) - \alpha(t)\|$, $t \in J$, Clearly, $z(t) = 0$ for $t_0 - r \le t \le t_0$. Now for $t \in I$, we obtain from (3.2), (3.3) and using (2.1) and (2.2),

$$z(t) = \| \beta(t) - \alpha(t)\|$$

$$\le \int_{t_0}^{t} \| f(s, \beta(s), \beta_s, \int_{t_0}^{s} k(s, \tau, \beta(\tau)\, \beta_\tau)\, d\tau)$$

$$- f(s, \alpha(s), \alpha_s, \int_{t_0}^{s} k(s, \tau, \alpha(\tau)\, \alpha_\tau) d\tau \| \, ds$$

$$\le M_3 \int_{t_0}^{t} \|\beta(s) - \alpha(s)\|_C\, ds + M_4 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds$$

$$+ M_1 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta(\tau) - \alpha(\tau) \|_C\, d\tau\, ds + M_2 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

$$\le M_3 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_4 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds$$

$$+ M_1 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds + M_2 M_5 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

$$= M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

where $M_6 = \max \{M_3 + M_4, (M_1 + M_2) M_5\}$. Thus $t \in I$, we have

$$(3.4) \quad \| \beta(t) - \alpha(t)\| \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

Case 1.

Suppose $t \ge t_0 + r$. Then for all $\theta \in [-r, 0]$, $t + \theta \ge t_0$. For such θ's from (3.4), we get

$$\|\beta(t+\theta) - \alpha(t+\theta)\| \leq M_6 \int_{t_0}^{t+\theta} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t+\theta} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

$$\leq M_6 \int_{to}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{to}^{t} \int_{to}^{s} \|\beta_\tau - \alpha_\tau\|_C \, d\tau \, ds$$

which yields

$$\text{(3.5)} \qquad sup_{\theta \in [-r,\, 0]} \|\beta(t+\theta) - \alpha(t+\theta)\| = \| \beta_t - \alpha_t \|_C$$

$$\leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

Case 2.

Suppose $t_0 \leq t < t_0 + r$. Then for all $\theta \in [-r, t_0, -t]$, we have $t + \theta < t_0$. we observe that

$$\text{(3.6)} \quad \| \beta(t + \theta) - \alpha(t + \theta)\| = 0$$

For $\theta \in [t_0 - t,\ 0], t + \theta \geq t_0$. Then, we get as in the case 1,

$$\text{(3.7)} \quad \| \beta(t + \theta) - \alpha(t + \theta)\|$$

$$\leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

Thus, for every $\theta \in [-r, 0]$, $(t_0 \leq t < t_0 + r)$, from (3.6) and (3.7), have

$$\| \beta(t + \theta) - \alpha(t + \theta)\| \leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

which yields

$$\text{(3.8)} \quad \| \beta_t - \alpha_t \|_C \leq M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C \, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C \, d\tau \, ds$$

For every $t \in I$, from (3.5) and (3.8), we have

$$(3.9)\quad \| \beta_t - \alpha_t\|_C \le M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

$$\le \varepsilon + M_6 \int_{t_0}^{t} \|\beta_s - \alpha_s\|_C\, ds + M_6 \int_{t_0}^{t} \int_{t_0}^{s} \| \beta_\tau - \alpha_\tau \|_C\, d\tau\, ds$$

where $\varepsilon > 0$ is an arbitrary constant. Now, by applying Lemma 2 with $u(t) = \| \beta_t - \alpha_t\|_C$ we get from (3.9)

$$\| \beta_t - \alpha_t\|_C \le \varepsilon\, [1 + M_6 \exp\{(M_6 + 1)T\} T]$$

Since ε is arbitrary, $\| \beta_t - \alpha_t \|_C = 0$, $t \,\varepsilon\, I$, which implies that $\| \beta(t) - \alpha(t)\| = 0$, $t \in I$ i.e. $\beta(t) = \alpha(t)$, $t \in I$. Hence,

$$\beta(t) = \alpha(t), \ t \in J$$

This complete the proof of the Theorem 2.

PROOFS OF THEOREMS 3 AND 4

The solution of the system (1.1) on $(t_0 - r, \infty)$ is given by

$$(4.1)\quad \begin{aligned} x(t) &= \phi(0) + \int_{t_0}^{t} f\Big(s, x(s), x_s, \int_{t_0}^{s} k(s, \tau, x(\tau), x_\tau)\, d\tau\Big)\, ds \qquad \text{for } t \ge t_0 \\ &= \phi(t - t_0), \qquad \text{for } t_0 - r \le t \le t_0 \end{aligned}$$

If $t \ge t_0$, then from (4.1) and using (2.3), (2.4), we get

$$(4.2)\quad \|x(t)\| \le \|\phi\|_C + \int_{t_0}^{t} L_2(s)[\| x\| + \| x_s\|_C]\, ds$$

$$+ \int_{t_0}^{t} L_2(s) \int_{t_0}^{s} L_1(\tau)\, [\| x\| + \| x_\tau\|_C]\, d\tau\, ds$$

$$\le \|\phi\|_c + \int_{t_0}^{t} 2L_2(s)\, \| x_s\|_c\, ds + \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\, \| x_\tau\|_C\, d\tau\, ds$$

By proceeding as in the proof of Theorem 2, we get from (4.2)

$$(4.3)\ \|x_t\|_C \leq \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\ \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\ \|x_\tau\|_C\, d\tau\, ds$$

Applying Lemma 2 with $u(t) = \| x_t\|_C$, we obtain from (4.3),

$$(4.4)\ \|x_t\|_C \leq \|\phi\|_C + [\ 1 + \int_{t_0}^{t} 2L_2(s)\exp\{ \int_{t_0}^{s} (2L_2(\tau) + L_1(\tau))\, d\tau \}\, ds$$

From (4.4) and (2.5), If follows that the solution $x(t)$ of the system (1.1) on $[t_o, \infty]$ is bounded. Consequently, the solution $x(t)$ of the system (1.1) is bounded on $[t_0 - r, \infty]$ and the proof of the Theorem 3 is complete.

Remark. It is to observe that our result proved in Theorem 3 yields the stability of the solution $x(t)$ of (1.1) if $\| \phi \|_C$ is small enough.

If $t \geq t_0$ then from (4.1) and making use of (2.6), (2.7), we obtain

$$(4.5)\ \|x(t)\| \leq \|\phi\|_C + \int_{t_0}^{t} e^{-\gamma s} L_2(s)[\|x\| + \|x_s\|_C]\, ds$$

$$+ \int_{t_0}^{t} e^{-\gamma s} L_2(s) \int_{t_0}^{s} e^{-\gamma(\tau - s)} L_1(\tau)\ [\|x\| + \|x_\tau\|_C]\, d\tau\, ds$$

$$\leq \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\ e^{-\gamma s}\ \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\ e^{-\gamma t}\ \|x_\tau\|_C\, d\tau\, ds$$

From (4.5) and by proceeding as in the proof of Theorem 2, we get

$$(4.6)\ \|x_t\|_C \le \|\phi\|_C + \int_{t_0}^{t} 2L_2(s)\, e^{-\gamma s} \|x_s\|_C\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) \int_{t_0}^{s} L_1(\tau)\, e^{-\gamma\tau} + \|x_\tau\|_C\, d\tau\, ds$$

Now, since $\gamma > 0$ and $t \ge t_0 \ge 0$, we have $e^{-\gamma t}\ 5\ e^{-\gamma t}\, 5\ 1$. Multiplying L.H.S. of (4.6) by $e^{-\gamma t}$ and R.H.S. of (4.6) by $e^{-\gamma t_0}$, we obtain

$$(4.7)\ e^{-\gamma t}\|x_t\|_c \le \|\phi\|_C\, e^{-\gamma t_0} + \int_{t_0}^{t} 2L_2(s)\ e^{-\gamma t_0}\ e^{-\gamma s}\|x_s\|_c\, ds$$

$$+ \int_{t_0}^{t} 2L_2(s) e^{-\gamma t0} \int_{t_0}^{s} L_1(\tau) e^{-\gamma\tau}\ \|x_\tau\|_C\, dr\, ds$$

Applying Lemma 2 with $u(t) = e^{-\gamma t}\|x_t\|_C$, the inequality (4.7) yields

$$(4.8)\ \|x_t\|_C \le \|\phi\|_C\ [1+\int_{t_0}^{t} 2L_2(s) e^{-\gamma t_0} \exp(\int_{t_0}^{s} \{2L_2(\tau) e^{-\gamma t_0} + L_1(\tau)\}\, d\tau)\, ds]$$

$$. e^{l(t-t_0)}$$

From (4.8) and (2.5), it follows that

$$\|x_t\|_C \ \ge Q\, \|\phi\|_C\, e^{\gamma(t-t)}, \qquad t \le t_0$$

where $Q > 0$ is a constant. Hence, the solution $x(t)$ of system (1.1) is uniformly slowly growing and the proof of the Theorem 4 is complete.

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ON ORTHOGONAL IDEALS IN GROUPRINGS

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ABSTRACT

In this paper we define orthogonal ideals of a groupring and we prove when the ring is Z_p, p - a prime and G - a cyclic group of order n, there exists a pair of orthogonal ideals I_1 and I_2 in Z_pG such that $Z_p G = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further we prove that these ideals are cyclic codes of dimension n-1 and 1 respectively.

DEFINITION 1

Let R be a ring. We say two ideals I and J are orthogonal if $I.J = (O)$ where $I.J = \{ i.j : i \in I, j \in J \}$.

The same definition holds for any grouping. Trivially every ideal is orthogonal to the (O) ideal.

Proposition 2

If R is a ring in which $R^2 = 0$, then every pair of ideals of R is orthogonal.

Proof: Obvious.

Proposition 3

If R is a ring in which $R^2 = 0$ and G is any group then in the grouping RG every pair of ideals is orthogonal.

Proof

As $R^2 = 0$ so $1 \notin R$. We have $RG.RG = (RG)^2 = RG = 0$. Hence by proposition 2, every pair of ideals in RG is orthogonal.

Example 1

$Z_2 = (0,1)$ and $G = \langle g : g^2 = 1 \rangle$. Then $Z_2G = (0, 1, g, 1+g)$. Let $I_1 = (0, 1+g)$, $I_2 = (0, 1+g)$,

$S = (0, 1, g)$. Here I_1 and I_2 are orthogonal ideals and S is a semigroup under multiplication. Further

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$$Z_2G = I_1 \cup I_2 \cup S$$

Example 2

Let $Z_2 = \{0,1\}$, $G = \{ g : g^3 = 1\}$ Then
$Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$
Let $I_1 = \{0, 1+g, 1+g^2, g+g^2\}$, $I_2 = \{0,1+g+g_2\}$
Clearly $I_1.I_2 = 0$. Further $Z_2G = I_1 \cup I_2 \cup S$ where
$S = \{0, 1, g, g^2, 1+g+g^2\}$ is a semigroup under multiplication.

Example 3

$Z_2 = \{0,1\}$, $G = \{g : g^4=1\}$
$Z_2G = \{0, 1, g, g^2, g^3, 1+g, 1+g^2, 1+g^3, g+g^2, 1+g+g^2, g+g^3, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3, 1+g+g^2+g^3\}$
Let $I_1 = \{ 0, 1+g, 1+g^2, 1+g^3, g+g^2, g+g^3, g^2+g^3, 1+g+g^2+g^3 \}$
$I_2 = \{ 0, 1+g+g^2+g^3 \}$

Clearly I_1 and I_2 are orthogonal. Now $Z_2G = I_1 \cup I_2 \cup S$ where
$S = \{0, 1, g, g^2, g^3, 1+g+g^2, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3 \}$ is a semigroup under multiplication.

Example 4:

$Z_3 = \{ 0, 1, 2 \}$, $G = \{ g : g^2 = 1 \}$
$Z_3G = \{ 0, 1, 2, g, 2g, 1+g, 2+g, 1+2g, 2+2g\} = I_1 \cup I_2 \cup S$, where $I_1 = \{ 0, 2+g, 2g+1\}$, $I_2 = \{0, 1+g, 2+2g\}$ and $S = \{0, 1, 2, g, 2g, 1+g, 2+2g\}$

Theorem 1:

Let $Z_p = \{ 0, 1, 2, ...,p-1\}$ and $G = \{ g : g^n = 1\}$.
Then in the groupring Z_pG there exists a pair of orthogonal ideals I_1 and I_2 such that $Z_pG = I_1 \cup I_2 \cup S$ where S is a semigroup under multiplication. Further $I_1 \cap S = \{0\}$. $I_2 \cap S = \{0\}$ if $p|n$ and $I_2 \subset S$ if $p \nmid n$.

Proof

$$Z_pG := \{\alpha = a_0 + a_1g + \dots + a_{n-1}\, g^{n-1} : a_i \in Z_p\}$$

$$\text{Let}\quad I_1 = \left\{ \alpha \in Z_pG : \sum_{1}^{n-1} a_i = 0 (mod\ p) \right\}$$

$I_2 = \{ k (g^{n-1} + g^{n-2} + \dots + g + 1), 0 \le k \le p-1\}$ and

$$S = \{0\} \cup \left\{ \alpha \in Z_p G : \sum_{i=1}^{n-1} a_i = 1 \text{ or } 2 \text{ or} \dots \text{ or } (p-1) \ (\mathrm{mod}\ p) \right\}.$$

Clearly I_1 and I_2 are ideals in $Z_p G$. Now suppose $\alpha \in I_1$ and $\beta \in I_2$.
Then $\alpha = a_0 + a_1 g + \dots + a_{n-1} g^{n-1}$, $\sum a_i = 0 \ (\mathrm{mod}\ p)$.
$\beta = k (g^{n-1} + g^{n-2} + \dots + g+1)$, $0 \le k \le p-1$.
Then $\alpha . \beta = (a_0 + a_1 + \dots + a_{n-1}). (k (g^{n-1} + g^{n-2} + \dots + g+1))$
$= \sum a_i . (k (g^{n-1} + g^{n-2} + \dots + g+1))$
$=0$, since $\sum a_i = 0 \ (\mathrm{mod}\ p)$.

Hence $I_1 . I_2 = \{0\}$. Therefore I_1 and I_2 are orthogonal.

Clearly S is a semigroup under multiplication. Also $Z_p G = I_1 \cup I_2 \cup S$. By the definition of I_1 and S we have

$I_1 \cap S = \{0\}$. Now if $p|n$, then $n = mp$, for some positive integer 'm'. Then $I_2 = \{k(1 + g + \dots + g^{mp-1}, 0 \le k \le p-1\}$. If $\alpha \in I_2$, we have
$\sum a_i = k..\ mp = 0 \ (\mathrm{mod}\ p)$. Therefore, $I_2 \subseteq I_1$. Hence $I_2 \cap S = \{0\}$.

If $p \nmid n$, then $p \nmid k.\ n$ for any k such that $1 \le k \le p-1$. Therefore, $\sum a_i \ne 0 \ (\mathrm{mod}\ p)$ for any $\alpha \in I_2$. That is, $\sum a_i = 1$ or 2 or ... or $(p-1) \ (\mathrm{mod}\ p)$. Thus $\alpha \in S$ which implies $I_2 \subseteq S$.

Theorem 2:

In $Z_p G$, where $G = \{ g;\ g^n = 1\}$, the ideal defined by $I_1 = \{ \sum_{i=0}^{n-1} a_i g^i :$ $\sum a_i = 0 \ (\mathrm{mod}\ p) \}$ is a cyclic code generated by "g-1".

Recall that a cyclic code of length n and dimension k is a principal ideal generated by a divisor of degree n-k of the polynomial g^n-1 in $Z_p G$.

Proof

Suppose $a_0 + a_1 g + \dots + a_{n-1} g^{n-1} \in Z_p G$. Then we have,
$(g-1).(a_0 + a_1 g + \dots + a_{n-1} g^{n-1})$
$= a_0 g + a_1 g^2 + \dots + a_{n-1} g^n - a_0 - a_1 g - \dots - a_{n-1} g^{n-1}$
$= (a_{n-1} - a_0) + (a_0 - a_1) g + \dots + (a_{n-2} - a_{n-1}) g^{n-1}$.

Here $\sum a_i = 0$. Hence the above element belongs to I_1.

Conversely, suppose $\sum_{i=0}^{n-1} b_i g^i \in I_1$. This implies that

$$\sum_{i=0}^{n-1} b_i = 0 \ (mod\, p)$$

We will find out an element $\sum_{i=0}^{n-2} a_i g^i$ of Z_pG such that

$$\sum_{i=0}^{n-1} b_i g^i = (g-1).(\sum_{i=0}^{n-2} a_i g^i)$$

For, consider the element $\sum_{i=0}^{n-2} a_i g^i$ where a_i's are given by

$$a_0 = -b_0,\ a_1 = -(b_0 + b_1),\ \ldots, a_{n-3} = -(b_0 + b_1 + b_{n-3}),$$
$$a_{n-2} = b_{n-1}$$

Now, $(g-1).(\sum_{i=0}^{n-2} a_i g^i) = (g-1).\{-b_0 - (b_0 + b_1)g - \ldots + b_{n-1} g^{n-2}\}.$

$$= b_0 + b_1 g + \ldots + b_{n-2} g^{n-2} + b_{n-1} g^{n-1},$$

since $\Sigma b_i = 0$ (mode p).

$$= \sum_{i=0}^{n-1} b_i g^i$$

Hence every element of I_1 is a multiple of $(g-1)$ and every multiple of $(g-1)$ belongs to I_1. This implies that I_1 is a cyclic code generated by $(g-1)$ and has dimension $n-1$.

Note : The ideal I_2 generated by $(g^{n-1} + g^{n-2} + \ldots + g + 1)$ is clearly the cyclic code of dimension 1.

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ON TRILATERAL GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS

Asit Baran Majumdar*

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ABSTRACT:

In this paper we derive a theorem on mixed trilateral generating functions of modified Laguerre polynomials by group theoretic method. A particular case of interest is also pointed out.

AMS classification code (1980) : 33A65.

Keywords and Phrases: Generating functions. Modified Laguerre Polynomials.

INTRODUCTION.

The modified Laguerre polynomial $f_n^{\beta}(x)$ is defined by [1]

$$f_n^{\beta}(x) = (-1)^n \; L_n^{-n-\beta}(x) = \frac{(\beta)_n}{n!} \; {}_1F_1\left[\begin{matrix} -n; \\ 1-\beta-n; \end{matrix} \; x\right] \tag{1.1}$$

In this note we prove the following theorem in connection with the trilateral generating relations of modified Laguerre polynomials from the Lie group view point.

Theorem : If there exists a bilateral generating relation of the form

$$G(x,u,t) = \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta}(x) \, g_n(u) \, t^n \tag{1.2}$$

where k is a non-negative integer and $g_n(u)$ is an arbitrary polynomial of degree n, then

$$\sum_{n=0}^{\infty} \psi_n(x,u,t) \, z^n = (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) \tag{1.3}$$

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where

$$(1.4) \qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

PROOF OF THE THEOREM

We shall first define the following linear partial differential operator for modified Laguerre polynomials.

$$(2.1) \qquad R = xyz^{-1}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} + yz^{-1}(1-x)$$

such that

$$(2.2) \qquad R\left(f_{n+k}^{\beta}(x)\, y^n z^{\beta}\right) = -(n+k+1)\, f_{n+k+1}^{\beta-1}(x)\, y^{n+1} z^{\beta-1}.$$

The extended form of the group generated by R is given by

$$(2.3) \qquad e^{wR} f(x,y,z) = \left(\frac{z}{z-wy}\right) \exp\left(\frac{wxy}{z-wy}\right) f\left(\frac{wz}{z-wy}.\, y,\ z-wy\right)$$

Consider the following formula

$$(2.4) \qquad G(x,u,t) = \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta}(x)\, g_n(u)\, t^n$$

where $g_n(u)$ is an arbitrary polynomial of degree n.
Replacing t by tz and then multiplying both sides of (2.4) by y^β, we get

$$(2.5) \qquad y^\beta G(x,u,tz) = \sum_{n=0}^{\infty} a_n \left(f_{n+k}^{\beta}(x)\, z^n y^\beta\right) g_n(u)\, t^n$$

Operating both sides of (2.5) by $exp(wR)$, we get

$$(2.6) \qquad exp\,(wR)\,(y^\beta G(x,u,tz))$$

$$= exp(wR) \sum_{n=0}^{\infty} a_n\, f_{n+k}^{\beta}(x)\, y^\beta\, g_n(u)\, (tz)^n$$

The left member of (2.6) becomes

$$(2.7)\qquad exp\left(\frac{-\dfrac{wxz}{y}}{1-\dfrac{wz}{y}}\right) y^{\beta}\left(1-\frac{wz}{y}\right)^{\beta-1} G\left(\frac{x}{1-\dfrac{wz}{y}}, u, tz\right)$$

Also the right member of (2.6) becomes

$$(2.8)\qquad \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^p}{p!} R^p\left(f_{n=k}^{\beta}(x)\, y^{\beta} z^n\right) g_n(u)\, t^n$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-w)^p}{p!}(n+k+1)_p\, f_{n+k+p}^{\beta-p}(x)\, z^{n+p}\, y^{\beta-p}\, g_n(u)\, t^n$$

$$= \sum_{p=0}^{\infty} z^n \sum_{p=0}^{\infty}\binom{n+k}{p} a_{n-p}\left(-\frac{w}{y}\right)^p f_{n+k}^{\beta-p}(x)\, y^{\beta}\, g_{n-p}(u)\, t^{n-p}$$

Then equating (2.7) and (2.8) we get

$$(2.9)\qquad \left(1-\frac{wz}{y}\right)^{\beta-1} exp\left(\frac{-\dfrac{wxz}{y}}{1-\dfrac{wz}{y}}\right) G\left(\frac{x}{1-\dfrac{wz}{y}}, u, tz\right)$$

$$= \sum_{n=0}^{\infty} \psi_n(x,u,t)\, z^n$$

where

$$(2.10)\qquad \psi_n(x,u,t) = \sum_{p=0}^{n}\binom{n+k}{p+k} a_p\left(-\frac{w}{y}\right)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

putting w/y =1 in (2.9), we get

$$(2.11)\qquad (1-z)^{\beta-1} exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right) = \sum_{n=0}^{\infty}\psi_n(x,u,t)\, z^n$$

where

$$(2.12)\quad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n+k}{p+k} a_p (-1)^{n-p} f_{n+k}^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

This completes the proof of the theorem.

Putting $k=0$, we get the following result of modified Laguerre polynomeals:

Corollary: If there exists a bilateral generating relation of the form

$$(2.13)\qquad G(x,y,t) = \sum_{n=0}^{\infty} a_n f_n^{\beta}(x)\, g_n(u)\, t^n$$

then

$$(2.14)\qquad (1-z)^{\beta-1} \exp\left(\frac{-xz}{1-z}\right) G\left(\frac{x}{1-z}, u, tz\right)$$

$$= \sum_{n=0}^{\infty} \psi_n(x,u,t) z^n$$

$$(2.15)\qquad \psi_n(x,u,t) = \sum_{p=0}^{n} \binom{n}{p} a_p (-1)^{n-p} f_n^{\beta-(n-p)}(x)\, g_p(u)\, t^p$$

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ROBUST ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

Gulab Singh*

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ABSTRACT

Gopalan and Dey [4] have developed a criterion, on the lines similar to one suggested by Box and Draper [2], for characterising the robust experimental designs assuming a fixed effect model or model - I (Eisenhart [3]). In this paper a criterion for characterising robust experimental row-column designs assuming random row and column effects has been developed and the robustness of the row column designs with random row and column effects has been examined.

Mathematics Subject Classification (1991) : *62K99*

Keywords : Random effect model, robustness, idempotent matrices, eigen values, outlier, equi-replicate, Kronecker product of matrices, heterogeneity, dispersion matrix, normal equations, incidence matrix.

INTRODUCTION

It is assumed that the analysis of variance model for the observations can be approximated by linear combinations (functions) of certain unobservable quantities known as 'effects'. The 'effects' not directly observable quantities are more of less idealised formulations of some properties of interest to the investigator in the phenomenon underlying the observations. The purpose of the analysis is to make inferences about some of the effects and these inferences to be valid regardless of the magnitudes of certain other effects, which may be present in the linear combinations. Each affect is regarded as either an unknown constant or else as a random variable. Following Eisenhart [3], if the effect is treated as an unknown constant, it is called a fixed effect model or model -I, otherwise it is called a random effect model or model - II. The equation expressing the observations as linear combinations of the effects is known as model equation. It is assumed that in every model equation there is at least one set of random effects equal in number to the number of observations, a different one

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of which appears in every observation. This is called the residual effect or the error term. Furthermore, there is usually one fixed effect, which appears in every model equation and is known as the general constant and is the mean in some sense of the observations.

A number of criteria have been enumerated by Box [1] for judging a good response surface design for fitting an empirical interpolation function $Y = X\beta + e$ where $Y = (y_1, y_2, \ldots, y_n)'$ is an nx 1 vector of response observations, X is an nxp matrix of predictor variable observations, β is a px1 vector of parameters to be estimated and e is an nx 1 vector of residuals. On of the criterion is that the design should be insensitive to the wild observations. Box and Draper [2] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [2] have shown that in order that the predicted response y at any given point be insensitive to the outlier, the quantity $r = \Sigma r_{uu}^2$ should be minimised, where r_{uu} is the u-th diagonal element of the matrix $R = X(X'X)^{-1}X$. The designs which minimise r are called *robust*.

2. ROW-COLUMN DESIGNS WITH RANDOM ROW AND COLUMN EFFECTS

We shall consider a row-column design in which v treatments are arranged in p rows and q columns, i-th treatment being replicated r_i times. The model assumed in this case will be

$$Y = \mu J + \pi \tau + L \phi + M \Phi + e$$

where $y = ((y_{ijk}))$ is the vector of response of the i-th treatment in j-th row and k-th column, μ is the general mean effect, $\tau = (\tau_1, \ldots, \tau_v)'$ is the vector of treatment effects, $\phi = (\phi_1 \ldots, \phi_p)$ is the vector of row effects, $\Phi = (\phi_1, \ldots, \phi_q)'$ is the vector of column effects, e is the vector of random errors, independently normally distributed with mean vector O and dispersion $\sigma^2 I$. Here ϕ and Φ are assumed to be vector of random effects distributed normally with mean vectors O and dispersion matrices $\sigma_r^2 I$ and $\sigma_c^2 I$ respectively. π', L' and M' are the incidence matrices of treatments versus observations, row versus observations and column versus observations respectively. I is an identity matrix and J is a vector with all elements as unity. The above model may be written as

$$Y = X\ 0 + e_1$$

where $e_1 = L\phi + M\phi + e$, $X = [J : \pi_{n,v}]$ and $\theta = (\mu,\tau)$ we have the dispersion matrix of e_1 as,

$$\begin{aligned} D(e_1) &= LL'\,\sigma_r^2 + MM'\,\sigma_c^2 + I\,\sigma^2 \\ &= Z^{-1}\,\sigma^2 \quad \text{(say)} \end{aligned}$$

where Z^{-1} is the variance-covariance matrix of the random components and is function of σ_r^2/σ^2 and σ_C^2/σ^2 which are assumed to be fixed and known. Application of usual least square technique to estimate the vector of parameters yields the normal equation,

$$\hat{\theta} = (X'Z^{-1}X)^{-} X Z^{-1} Y$$

where $(X'Z^{-1}X)^{-}$ is generalised inverse (g - inverse) of $X'Z^{-1}X$ satisfying $X'Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}X = X'Z^{-1}X$. It is well known that in the absence of any outlier $R_0^2/(n-m)$ is an unbiased estimator of σ^2 where,

$$R_0^2 = Y'\,[Z^{-1} - Z^{-1}X\,(X'Z^{-1}X)^{-}X'Z^{-1}]\,Y$$

m = rank (X) and n is the number of observations.

When u-th observation has added to it a quantity c making it an outlier, the bias or discrepancy in estimating σ^2 through R_0^{*2} is $d_u = c^2 a_{uu}/(n-m)$ where R_0^{*2} is the residual sum of squares in the presence of the outlier and a_{uu} is the u-th diagonal element of the matrix

$$\begin{aligned} A &= Z^{-1} - Z^{-1}X\,(X'Z^{-1}X)^{-}X'Z^{-1} \\ &= Z^{-1} - D\,(Z^{-1}X'\,\hat{\theta}) \end{aligned}$$

where D stands for the dispersion matrix. If it were equally likely that c could occur with any of the n observations, giving rise to $d_1,\ldots,d_n$, the average discrepancy would be

$$\bar{d} = c^2 \sum_u a_{uu} / n(n-m)$$

which obviously is not fixed, unlike in the case of the fixed effect model, but is design dependent through A. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, the average discrepancy should be minimised. A design minimising the average discrepancy, in case of a random effect model, may be called *robust*.

We have,

$$D(Z^{-1}X'\hat{\theta})/\sigma^2 = Z^{-1}\pi'\,(\pi Z^{-1}\pi')^{-1}\,\pi' Z^{-1}$$

For getting the average discrepancy we need

$$\begin{aligned} trace\ Z^{-1}\pi'(\pi Z^{-1}\pi')^{-1}\pi Z^{-1} &= trace\ (\pi Z^{-1}\pi')^{-1}\pi Z^{-1}Z^{-1}\pi' \\ &= trace\ (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi' \end{aligned}$$

We examine below, robustness of row-column designs based on the criterion discussed above.

3. ROBUSTNESS OF ROW-COLUMN DESIGN

In case of a row-column design in v treatments, p rows and q columns the form of the matrix **Z** will be,

$$\mathbf{Z} = \begin{bmatrix} (1+\delta_2)+\delta_1 J & \delta_2 I & \ldots & \delta_2 I \\ \delta_2 I & (1+\delta_2)+\delta_1 J & \ldots & \delta_2 I \\ \vdots & \vdots & & \\ \delta_2 I & \delta_2 I & \ldots & (1+\delta_2)+\delta_1 J \end{bmatrix}$$

$$\text{or } Z = I_p \otimes [I_q + \delta_1 J_q] + J_p \otimes \delta_2 I_q$$

$$= I_p \otimes I_q + \delta_1 I_p \otimes J_q + \delta_2 J_p \otimes I_q$$

where $A \otimes B$ denotes the Kronekcer product of matrices A and B. J_n is an nxn matrix with all elements as unity, $\delta_1 = \sigma_r^2/\sigma^2$ and $\delta_2 = \sigma_c^2/\sigma^2$. The form of Z^{-1} will be

$$Z^{-1} = a_{oo} I_p \otimes I_q + a_{01} I_p \otimes J_q + a_{10} J_p \otimes I_q + a_{11} J_p \ominus J_q$$

where ,

$$a_{oo} = 1$$
$$a_{01} = -\delta_1/(1+q\delta_1)$$
$$a_{10} = -\delta_2/(1+p\delta_2)$$
$$a_{11} = -\delta_1\delta_2(2+p\delta_2+q\delta_1)/[(1+p\delta_2)(1+q\delta_1)(1+p\delta_2+q\delta_1)$$

In order to compute the average discrepancy in estimating σ^2 through R^{*2}_0 in the presence of an outlier. we need trace $(\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi'$. For a row-column design we have,

$$\pi Z^{-1}\pi' = \pi[I_p \otimes I_q + a_{01} I_p \otimes J_q + a_{10} J_p \otimes I_q + a_{11} J_p \otimes J_q]\pi'$$

$$= r^d + a_{01} N_1 N_1' + a_{10} N_2 N_2' + a_{11} rr'$$

and

$$\pi Z^{-2}\pi' = r^d + b_{01} N_2N_1' + b_{10}N_2N_2' + b_{11}rr'$$

where

$$b_{01} = -\delta_1 (2 + q\delta_1) / (1 + q\delta_1)^2$$

$$b_{10} = -\delta_2 (2 + p\delta_1) / (1+p\delta_2)^2$$

$$b_{11} = 2 (1 + p\delta_2 + q\delta_1) - pq(p\delta_2 + q\delta_1)\ \delta_1\delta_2 / [(1+p\delta_2 + q\delta_1)(1+p\delta_2)(1+ q\delta_1)]$$

and r^d is a diagonal matrix with elements of the replication vector r in the diagonal elements, N_1 and N_2 are the treatment versus rows and treatment versus column incidence matrices respectively.

We shall consider binary equi-replicate row-column designs for which N_1N_1' and N_2N_2' commute. For such designs

$$\text{trace}\quad \left(\pi Z^{\,\prime}\pi'\right)'\left(\pi Z^2\pi'\right) = \frac{r + b_{01}vr + b_{10}vr + b_{11}vr^2}{r + a_{01}vr + a_{10}vr + a_{11}vr^2} + \sum_{i=1}^{v-1} \frac{r + b_{01}\theta_{1i} + b_{10}\theta_{2i}}{r + a_{01}\theta_{1i} + a_{10}\theta_{2i}}$$

where θ_{1i} and θ_{2i} are the *i-th* eigen values of the matrices N_1N_1' and N_2N_2' respectively. From equation (3.1) it is clear that the average discrepancy is not fixed, as in the case of fixed effect model, but is design dependent through θ_i's. In order that the average discrepancy is minimum, the trace given in (3.1) should be maximum subject to the condition that sum of all the eign values is constant. So maximising (3.1) subject to the condition that $\sum_{i=1}^{v-1}\theta_{1i} = r(v-k)$ and $\sum_{i=1}^{v-1}\theta_{2i} = r(v-k)$, we may see that all θ_{1i}'s are same and so are θ_{2i}s. Let $\theta_{1i} = \theta_1$ and $\theta_{2i} = \theta_2$ so that $\theta_1 = r(v-q) / (v-1)$ and $\theta_2 = r(v-p) / (v-1)$

The characteristic vector corresponding to the eigen value rq is $rq\ JJ'$ and the eigen vector corresponding to the eigen value $r(v-q)/(v-1)$ is $(I-JJ'/v)$, so we have,

$$N_1N_1' = rq\ JJ'/v + r(v-q)\ (v-1)^{-1}\ (I - JJ'/v)$$

similarly, $\quad N_2N_2' = rp\ JJ'/v + r(v-p)\ (v-1)^{-1}\ (I - JJ'/v)$

giving,

$$pN_1N_1' + qN_2N_2' = r\ (v-1)^{-1}\ [\{ v (p + q) - 2pq \}\ I - \{p + q - 2pq\}\ JJ']$$

which is satisfied by balanced incomplete row-column designs (Singh [5]), showing that balanced incomplete row-column designs with random row and column effects are robust.

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INTER BLOCK ANALYSIS OF BALANCED INCOMPLETE BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

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ABSTRACT

Balanced incomplete block designs with nested rows and columns (BIBRC) were introduced by Singh and Dey [2] as generalisation of the Lattice Square designs. The intra-block analysis of these designs has been developed by Singh and Dey [2]. Since the blocks of the BIBRC are incomplete, assuming the block effects as random and row, column classification complete, an attempt has been made in this paper to develop the inter block analysis of these designs.

Mathematics Subject Classification (1991) : 62K10

Keywords : Lattice Square designs, heterogeneity, orthogonal, dispersion matrix, Kronecker product of matrices, maximum likelihood estimates, random effect model.

INTRODUCTION

The experimental units within a complete block are often subjected to more than one source of variation. Recognising the need for appropriate experimental designs for such situations Yates [3,4] developed a group of block designs with possibility of eliminating heterogeneity in two directions within the complete block. Singh and Dey [2] defined and studied a class of experimental designs which are arrangements of treatments in several sets (blocks) of rows and columns such that within each set (block) the row versus column classification is orthogonal, though the sets themselves are incomplete. These designs may be viewed as the generalization of the Lattice Square

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designs of Yates [4]. Designs defined by Singh and Dey [2] are different from the designs of Preece, Pearce and Kerr [1] who consider designs with three mutually orthogonal and fully crossed classifications while Singh and Dey [2] consider designs with a set of blocks (sets) within which are nested two more classifications, rows and columns.

NOTATIONS AND FORMULATION OF THE MODEL FOR ANALYSIS

Let us suppose that v treatments have been allotted to s blocks (sets) and the experimental units within each set are further arranged in p rows and q columns, enabling to control the variability within a set, in two directions. The model assumed in this case is

$$y_{i_{(j_m j_l)}} = \mu + \Omega_j + \phi_{j_m} + \delta_{j_l} + \tau_i + \varepsilon_{i_{(j_m j_l)}} \tag{1}$$

where, $y_{i_{(j_m j_l)}}$ is the response obtained from the unit receiving i-th treatment in the $(j_m j_l)$-th cell of the j-th set, μ is the general mean effect, Ω_j is the effect of the j-th set (block) ($l = 1, \ldots\ldots, s$), ϕ_{j_m} is the effect of m-th row of the j-th set ($m = 1, \ldots.., p$), δ_{j_l} is the effect of the l-th column in the j-th set ($l = 1, \ldots., q$), τ_i is the effect due to i-th treatment ($i = 1, \ldots., v$) and $\varepsilon_{i_{(j_m j_l)}}$'s are uncorrelated residuals distributed normally with mean zero and constant variance σ^2. Since model (1) is assumed as random effect model, we further assume the following distributional properties of different effects i.e. δ_{j_l} is distributed normally with mean zero and variance $\sigma^2_{c_j}$, ϕ_{j_m} is distributed normally with mean zero and variance $\sigma^2_{r_j}$ and Ω_j is distributed normally with mean zero and variance σ_s^2. Also ϕ_{j_m}, δ_{j_l}, and $\varepsilon_{i_{(j_m j_l)}}$ are independently distributed. further,

$$V[\, y_{j(m,l)} \,] = \sigma^2 + \sigma^2{}_s + \sigma^2{}_{r_j} + \sigma^2{}_{c_j}$$

$$Cov[\, y_{j(m,l)}, y_{j(m',l)} \,] = \sigma^2{}_s + \sigma^2{}_{c_j} \qquad , m \neq m'$$

$$Cov[\, y_{j(m,l)}, y_{j(m,l')} \,] = \sigma^2{}_s + \sigma^2{}_{r_\partial} \qquad , l \neq l'$$

$$Cov[\, y_{j(m,l)} \,,\, y_{j(m',l')} \,] = \sigma^2{}_s \qquad , l \neq l' \,, m \neq m'$$

$$Cov[\, y_{j(m,l)} \,,\, y_{j(m,l)} \,] = 0 \qquad , j \neq j'$$

Let $y_{j\,(m,.)}$, $y_{j(\,.\,,.)}$ and $y_{.(.\,,\,.)}$ denote the total of the 1-th column of the j-th set, total of the j-th set and grand total respectively. Further let us denote the vectors of the column totals of the j-th set, totals of the set (block) and grand total respectively as,

$$Y_{j(m,)} \qquad = [\, y_{j(m,1)} \,,\, \ldots\ldots\ldots\ldots,\, y_{j(m,q)} \,]'_{qx1}$$

$$Y_{j(.,)} \qquad = [\, y'_{j(1,)} \,, \ldots\ldots\ldots\ldots\ldots,\, y'_{j(m,)} \,]'_{pqx1}$$

$$Y_{.(.,)} \qquad = [\, y'_{1(.,)} \,, \ldots\ldots\ldots\ldots\ldots,\, y'_{s(.,)} \,]'_{spqx1}$$

it may easily be verified that

$$D\left[Y_{j(m,)}\right] = (\sigma^2 + \sigma^2{}_{c_j})I + (\sigma^2{}_s + \sigma^2{}_{r_j})J_{qq}$$

$$Cov\left[Y_{j(m,)}, Y_{j(m',\,)}\right] = \sigma^2_{cj}\, I + \sigma^2_s\, J_{q,q} \qquad m \neq m'$$

$$Cov\left[Y_{j(m,)}, Y_{j'(m',)}\right] = 0_{q,q} \qquad j \neq j',\, m \neq m'$$

where **D**(.) stands for the dispersion matrix, I is the identity matrix and J is the matrix with all elements as unity. Thus,

$$D[y_{j(m,)}] = \sigma^2 I_{pq} + \sigma^2_{r_j}\, I_p \otimes J_{q,q} + \sigma^2_{c_j} J_{p,p} \otimes I_p + J_{qp,\,pq}$$

$$V_j\,(say)\; j = 1,\, \ldots,\, s$$

$$\begin{aligned} D[Y_{.(.,.)}\,] &= diag(V_1,\, \ldots, V_s) \\ &= I_s \otimes (\sigma^2 I_{pq} + \sigma_s{}^2 J_{pq,qp}) + diag\,(\sigma^2_{r_1},\, \ldots, \sigma^2_{r_s}) \otimes (I_p \otimes J_{q\text{-}p}) \\ &\quad + diag\,(\sigma^2_{c_1}\,,\, \ldots,\, \sigma^2_{c_s}) \otimes (J_p \otimes I_p) \qquad (2) \\ &= V\,(say) \end{aligned}$$

Where $\otimes$ stands for Kronecker product of matrics. Under these assumptions, the joint distribution of the elements of the response vector y is given by

$$f(y, \theta) = [(2\pi)^{n/2} |V|^{1/2}]^{-1} \exp^{-1/2} (y-\mu J - \Delta'\tau)' V^{-1} (y-\mu J - \Delta'\tau)$$

Where $\theta' = (\mu, \sigma^2, \sigma^2_s, \sigma^2_{r_j}, \sigma^2_{c_j})$, $\Delta_{v,m}$ is treatment versus observation incidence matrix and $|V|$ is the determinant of V, From (2) we have,

$$V^{-1} = diag\,(V_1^{-1}, V_2^{-1}, \ldots, V_s^{-1})$$

with $\quad V_j^{-1} = a_j I_p \otimes I_q + b_j I_p \otimes J_p + c_j J_{p,p} \otimes I_q + d_j J_{p,p} \otimes J_{q,q}$

where a_j, b_j, c_j and d_j are functions of σ^2, $\sigma_s^2 \sigma^2_{c_j}$ and $\sigma^2_{r_j}$ given by

$$a_j = 1/\sigma^2$$

$$b_j = -\rho^2_{r_j} / \{(1+q\rho^2_{r_j}) \sigma^2\}$$

$$c_j = -\rho^2_{c_j} / \{(1+q\rho^2_{c_j}) \sigma^2\}$$

$$d_j = -\frac{\rho^2_{r_j} \rho^2_{c_j} (2 + p\rho^2_{c_j} + p\rho^2_{r_j} + pq\rho^2_s) - \rho^2_s}{\sigma^2 (1 + p\rho^2_{c_j})(1 + \rho^2_{r_j})(1 + p\rho^2_{c_j} + q\rho^2_{r_j} + pq\rho^2_s)}$$

where $\rho^2_{r_j} = \sigma^2_{r_j}/\rho^2$, $\rho^2_{c_j} = \sigma^2_{c_j}/\sigma^2$ and $\rho_s^2 = \sigma_s^2/\sigma^2$

Writing model (1) as $y = X\beta$, where β is the vector of parameters, maximum likelihood estimates of the vector of parameters may be obtained by solving the normal equations.

$$\left(\sum_{j=1}^{s} X_j' V_j^{-1} X_j\right)\beta = \sum_{j=1}^{s} X_j' V_j^{-1} y_j \tag{3}$$

The left hand side of equation (3) may be simplified further. Let α_{ij} denote the incidence vector of the i-th treatment in the j-th set. Obviously, we have

$$\alpha'_{ij}\alpha_{ij} = r_j$$
$$\alpha'_{ij}\alpha_{i'j} = 0 \qquad i \neq i' \text{ and}$$
$$J'\alpha_{ij} = r_{ij}$$

r_{ij} being the number of times the i-th treatment has been replicated in the j-th set, r_j is the vector of replication of treatments in the j-th set and $J = (1, 1, \ldots\ldots, 1)'$, Model (1) may also be written as,

$y_i = \mu J + \tau +$ error components of sets, rows within set and columns within set and the residual

$$E(y_j) = \left[J_{pq,1} : \mathbf{a}_{ij}, \ldots, \mathbf{a}_{vj}\right] \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_v \end{bmatrix}$$

$$= X_j \beta$$

Using these notations it may easily be verified that the term on the left hand side of equation (3) is

$$\left(\sum_{j=1}^{s} X'_j V^{-1}{}_j X_j\right) = \begin{bmatrix} J'V^{-1}J & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{1j} & \sum_{j=1}^{s} J'V_j^{-1}\alpha_{vj} \\ & \sum_{j=1}^{s} \alpha'_{ij} V_j^{-1}\alpha_{ij} & \sum_{j=1}^{s} \alpha'_{ij} V_j^{-1}\alpha_{ij} \end{bmatrix}$$

The general solutionof the normal equation(3) is rather complicated. However, for a particular design and for various values of the ratios, ρ_{r_j}, ρ_{i_j} and ρ_s the inter block estimates of the parameters may be obtained by solving the aforesaid normal equations.

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HINDU TRIGONOMETRY†

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ABSTRACT

In this paper we attempt to give a historical back ground of trigonometry and some trigonometrical formulae form *siddhanta Siromaṇi*. The conclusion is that the trgonometrical functions travelled from India to Europe via Arab.

Mathematics Subject Classifications (1991). : 01A32

Keywords & phrases : Trigonometrical functions, Hindu sine/cosine/versed sine.

THE ORIGIN OF TRIGONOMETRICAL FUNCTIONS AND THEIR HINDU NAMES

JYĀ

The fundamental term "*jyā*" of trigonometry appears in *Ṛgveda* (6.75.3).

योषे॑व शि॒ङ्क्ते॒ वित॒ताधि॒ धन्व॒ञ्ज्या इ॒यं सम॑ने पा॒रय॑न्ती ।
अहि॑रिव भो॒गैः पर्ये॑ति बा॒हुं ज्यायां॑ हे॒तिं प॑रि॒बाध॑मानः ॥

"*Jyā*" is the original word of Sanskrit language. The literal meaning of *jya* is the string of the *dhanusa* (bow) (See also [10, p. 197) and [17, p. 24].

"*Jyā*" also appears in *Sūrya Siddhānta*. We quote the following (cf. [7], [9] and [14].

राशिलिप्ताष्टमो भागः प्रथमं ज्यार्धमुच्यते ॥
तत्तद्विभक्तलब्धोनमिश्रितं तद् द्वितीयकम् ॥
आद्येनैवं क्रमात् पिण्डान् भक्त्वा लब्धोनसंयुताः ॥
खण्डकाः स्युश्चतुर्विंशज्ज्यार्धपिण्डाःक्रमादमी ॥

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Indeed trigonometry forms the base of *Sūrya Siddhānta*, and it developed in India as the stern daughter of astronomy.

A special name for the function *jyā* which we call the sine also appears in the *Ganita - Pada* of *Āryabhaṭiya* of *Āryabhaṭa* (b. 476 A.D.). However instead of *jyā*, he used *ardhajyā* (half chord). A (Hindu) sine table has also been given by *Āryabhaṭa* (see [14, p. 49]).

The word *jya* also appears in *Pnacasiddhāntikā* of *Varahamihira* (b. 505 A.D.) (See [15]). He also formed a (Hindu) sine table with a diameter of 120.

The term *jyā* formed its way into the works of *Brahmagupta* as *kramajyā* i.e. a straight line or *sinus rectus*.

Now the question arises how this word *jyā* is changed to sine. When this word transferred from India to Arab, it was pronounced *jaib* (see Gingerich [4], Smith [13] and Srinivasiengar [14]). And Arabians used this unrelated word. In fact *jaib* means bosom, breast and bay. When Gherardo of Cremona (1150 A.D.) made his translation from the Arabian literature, he used *sinus* for *jaib*. *Sinus* means bosom, bay, a curve, the fold of the toga about the breast, the land about a gulf and a fold in a land.

The word *kramajyā (jyā)* changed into *karaja* and *kardaja* at the hands of Arabians.It appears in the Bagdad school in the 9th century. In particular, *Al-Khowārizmī* used in the extract, which he made from the *Brahmasphutasiddhanta* of *Brahmagupta*. This work is known *Sindhind*. This word *jyā* also appears in the writings of the Spanish *Arab-ibn-Al-Zargāla* (1050 A.D.).

ABBREVIATION OF *JYĀ* (SINE)

The first mathematician who abbreviated sine into sin was Girard (1626 A.D.), (cf. [13]).

KOTIJYĀ (COSING)

Cosine is the modern name of Sanskrit word *kotijyā* that was used by ancient Indian astronomers. It appears in *Sūrya Siddhānta* and in the works of *Āryabhaṭa*. *Koti* means the (perpendicular) side of a right triangle, but it also means the end of *dhanuṣa* (bow). In other words *koṭijyā* means complementary *jya*

ABBREVIATION FOR *KOṬIJYĀ* (COSINE)

Plato of Tivoli (1120 A.D.) used *chorda residui* or the complement angle .Regiomontanus (1463 A.D.) used *sinus rectus* complementi. Rhaeticus (1515 A.D.) preferred basis. Vieta (1579 A.D.) used *sinus residuae*. Magini (1609 A.D.) used *sinus secundes*. Edmud Gunter (1620 A.D.) suggested *co-sinus* John Newton (1658 A.D.) used *cosinus* and this word received general favour. Jonas Moore (1674 A.D.) used cos and this symbol generally has been adopted by later writers (cf. Smith [13]).

UTKRAMAJYĀ (VERSED SINE)

The function *utkramajyā* first appears in *Surya Siddhānta* (280-400 A.D.) and immediately in the writings of *Āryabhata* I. It is also known as *ujjayā*. *Bhāskarācārya* in his *Lilāvati* calls it *Śara* (see below). It is (now) denoted by $Hvers\,Sin\theta (Hvers\ \theta) = R - H\cos\theta$, wherein R is the radius of the circle. In modern language it is denoted by $Versin\theta = 1 - \cos\theta$ (see Smith [13, part II, p. 618] and Kumari [7, Part I, p. 117]) From India it passed to Arab and they called it turned chord that is the arc of a circle.

THE VALUES OF *JYĀ* AND *KOṬIJYĀ* FUNCTIONS

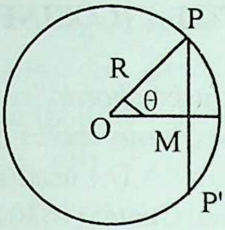
Munjāla (932 A.D.) defined the positive and negative values of the functions *jyā* and *Kotijyā* (see for instance, [16,p. 250]), in the following verse of his *Laghumānasa*

ग्रहः स्वोच्चोनितः केन्द्र तदूर्ध्वाधोऽर्धजो भुजः ।
धनर्ण पदशः कोटी घनर्णर्ण धनात्मिका ॥

This means :

In the upper semicircle (1-2 quadrants), the value of the function *jyā* remains positive and the function *kotijyā* remains +,-,-,+ in the four quadrants respectively.

HINDU DEFINITION OF *JYĀ* (SINE), *KOṬIJYĀ* (COSINE) AND *UTKRAMAJYĀ* OR *ŚARA* (VESED SINE)



In figure 1, 0 is the centre of the circle, PP' is the full chord of the circular arc PAP' and A is the middle point of the arc PAP'. On account of its shape PAP' is called the *Cāpa* . Let the arc PA subtend an angle θ at the cente 0. In Hindu trigonometry the angle θ is connoted by the corresponding arc AP of the circle. The chord PP' is the string of the bow and is called *jyā* in Sanskrit. PM which is half of the full chord PP' is known as (*Ardhyajyā* or simply *jyā*). Thus

$$Jya\ \widehat{PA} = PM = Hsin\,\theta$$

$$Kotijya\ \widehat{PA} = OM = Hcos\ \theta$$

$$Versed\ sine\ \widehat{PA} = MA = Hvers\,\theta$$

where PA stands for the arc PA and H for Hindu. Thus

$Hsin\,\theta = Rsin\,\theta$, $Hcos\,\theta = Rcos\,\theta$ and
$Hvers\,\theta = Rvers\,\theta = R\text{-}Hcos\,\theta$ (see [1],[2],[5] and [8]).

VALUE OF RADIUS R

The circumference of the circle is divided into 360 x 60 = 21600 equal parts and each part is taken to be a unit length, i.e. *Kalā* (minute).

Thus $2\pi R = 21600$
$R = 21600 \div 2\pi$, where $\pi = 3.1416$

The radius R is generally supposed to be 3438' (see [1], [2], [3], [7], [11], [12] and [16]. Recently, Hogendijk [6] has illustrated that R is half the circumference of the circle, and that

$R = (180/\pi)^\circ = (10800/\pi)' = 3437.7' \cong 3438'$

He [op. cit.] further says, "Sine tables with the base $R = 3438'$ are of (pre-Islamic) Indian origin and were known in the early Islamic tradition, although no such tables survive in the Arabic sources".

Sometimes the radius is taken to be 120 units, and *Brahmagupta's Laghujyā* is indeed Hsine in this case. *Bhāskarācārya* also uses these *Laghujyas* in the rectification of planets. *Bhāskarācārya takes* the $R = 3437.4766$ units (see [1] which is evidently a slightly better than 3438 units.

5. SOME BASIC FORMULAE

It is obvious from the right triangle POM,
(1) $Hsin^2\theta + Hcos^2\theta = R^2$
In modern language,

$$R^2 sin^2\theta + R^2 Cos^2\theta = R^2$$

which gives the well - known formula of trigonometry :

$$sin^2\theta + cos^2\theta = 1$$

Bhāskarācārya writes in *Golādhyāya* ;

> क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धांशकशिञ्जिनी स्यात् ।
> त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धांशकर्शिञ्जिनी वा ॥
> तस्या: पुनस्तद्दलभागकानां कोटेश्च कोट्यंशदंलस्य चैवम् ।
> अन्यज्यकासाधनमुक्तमेवं पूर्वै: प्रवक्ष्येऽथ विशिष्टमस्मात् ॥

Thus according to Somayaji [2] means :

(2) $Hsin\ \theta/2 = \frac{1}{2}\ (Hsin^2\theta + Hvers^2\ \theta)^{1/2}$

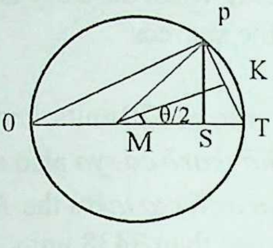
(3) $Hsin\ \theta/2 = (\frac{1}{2}\ R \times Hvers\ \theta)^{1/2}$

We give a proof of these formulae,

Proof. In figure 2, let $\angle PMT = \theta$, PS $\perp$ OT, K the middle point of PT and R

the radius of the circle OPT. Then

$$KMT = \frac{\theta}{2} \text{ and } TK = \frac{1}{2}\ PT$$



The following steps are evident from the right triangle PST.

$$PT^2 = PS^2 + ST^2$$

$$PT^2 = Hsin^2\theta + Hvers^2\theta$$

$$\text{and } Hsin\ \frac{\theta}{2} = TK = \frac{1}{2}PT = \frac{1}{2}(Hsin^2\theta + Hvers^2\theta)^{1/2}$$

From the right triangle POT,

$$PT^2 = TS \times OT$$

$$PT^2 = (Hvers\ \theta) \times 2R \text{ (since } OT = 2R)$$

$$PT = (2R \times Hvers\ \theta)^{1/2}$$

$$2Hsin\ \frac{\theta}{2} = (2R \times Hvers\ \theta)^{1/2}$$

$$Hsin\ \frac{\theta}{2} = \frac{1}{2}(2R \times Hversa\ \theta)^{1/2} = (\frac{1}{2}\ R \times Hvers\theta)^{1/2}$$

Bhāskarācārya has given several other trigonometrical formulae. The verse numbers (12), (13), (14), (15), (16), (17), (18), (19), (20), (21) and (22) of *Golādhyāya* (*jyot-pattivāsanā*) give the following formulae :

$$(4)\ \ Hsin\frac{(90^o\ +\ x)}{2} = [1/2(R^2 + RHsin\ x)]^{1/2}$$

$$5)\ \ \sqrt{2}\ Hsin\left(\frac{x\ -\ y}{2}\right) = [1/2\{(Hsinx - Hsiny)^2 + (Hcosx - Hcosy)^2\}]^{1/2}$$

(6) $[(Hcosx - Hsinx)^2]^{1/2} = Hsin(45^\circ - x)$

(7) $R - (2Hsin^2x)/R = Hsin(90^\circ - 2x)$

(8) $Hsin(x \pm 1) = Hsinx\ (1 - 1/6569) \pm 10/573\ Hcosx$

(9) $H_{r+1} = H_r(1 - 1/476) + Hcosx_r\ (100/1529)$

(10) $Hsin(x + y)\ (Hsinx.\ Hcosy + Hcosx.Hsiny)\ \div R$

For details refer to Somayaji [1] and [2].

For the proof of the formula (8) Somayaji has suggested the interpolation method. We give below a somewhat straight proof.

To prove (8), we note that

$Hsin(x+1)^\circ = (Hsinx^\circ.Hcos1^\circ)/R \pm (Hsin1^\circ.Hcosx^\circ)/R$

Now, put $Hsin1^\circ = 60'$, $Hcos1^\circ = 3437.4766$ and

$R = 3438'$ to get,

$$\begin{aligned} Hsin(x\pm1)^\circ &= (Hsinx^\circ.3437.4766/3438) \pm (60Hcosx^\circ/3438) \\ &= Hsinx\ (1 - .5234/3438) \pm (10Hcosx/573) \\ &= Hsinx(1 - 1/6568.5899) \pm (10/573)\ Hcosx \\ &= Hsinx\ (1 - 1/6569) \pm (10\ Hcosx)/573 \end{aligned}$$

For the solution of diurnal problems the verse numbers (54), (55) and (57) of *Grhaganita* (*Spastādhikāra*) give the following formula :

(11) $Hsin\ \alpha = R(Hsin^2\lambda - Hsin^2\ \delta)^2 + Hcos\ \delta$

(12) $Hsin\ \alpha = Hsin\ \lambda \times Hcos\,\omega \div Hcos\ \delta$

Wherein λ, δ, α and ω are the longitude declination, right ascensian of the sun and obliquity of the ecliptic respectively. For details refer to Somayaji [1] and [2]

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ON CONVERGENCE AND FIXED POINTS OF SEQUENCES OF MAPPINGS

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ABSTRACT

Two results on common fixed points have been obtained. The first theorem deals with convergence of the sequence of common fixed points of four sequences of selfmappings on a metric space. In the second result an existence theorem for common fixed points of two sequences of mappings with two selfmappings has been established.

Mathematics subject classification (1991) : 54H25, 47H10

Key words and phrases : Weakly commuting mappings, sequences of mappings, common fixed point

INTRODUCTION

Several authors have dealt on the question of convergence of a sequence of fixed points or commom fixed points of mappings and many interesting results have been established. In the present paper we prove a theorem which gives sufficient conditions under which the uniform convergence of four sequences of mappings to four given mappings satisfying a Meir and Keeler type contractive condition implies the convergence of the common fixed points of the sequences of mappings to a common fixed point of the four mappings. The second theorem is an extension, to the case involving sequences of mappings, of the following recent theorem by the first author [8,p.78], [9]:

Theorem A Let P and S be weakly commuting mappings and Q and T be weakly commuting mappings of a complete metric space $(X.d)$ into itself satisfying the conditions:

Given $\varepsilon > O$, there exists an $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of ε such that for all x, y in X

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$$\varepsilon \leq \max\{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty),$$

$$[d(Px,Ty) + d(Qy,Sx)]/2\} < \varepsilon + h$$

$$\Rightarrow d(Px,Qy) < \varepsilon \qquad \text{.......(i)}$$

$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. ....(ii)

Let the range of T contain the range of P and the range of S contain the range of Q. If one of the mappings P, Q, S and T is continuous then the mappings P,Q, S and T have a unique common fixed point which is also the unique common fixed point of P and S and of Q and T.

It is of interest to note that in the present extension of the above theorem we have assumed very few additional conditions.

Both of our results extend and unify several important fixed point theorems.

RESULTS

Two selfmappings F and G of a metric space (X,d) are called weakly commuting provided

$$d(FGx,\ GFx) \leq d(Fx,Gx)$$

for all x in X.

Theorem 1 Let P_n, Q_n, S_n and T_n be mappings of a metric space (X,d) into itself with at least one common fixed point u_n for each n = 1,2,... . Let P, Q, S and T be selfmsappings of X having a common fixed point u and satisfying the conditions:

Given $\varepsilon > O$, there exists $h(\varepsilon) > O$, $h(\varepsilon)$ being a nondecreasing function of $\in$, such that for all x,y in X

$$\varepsilon \leq \max\{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty).\ [d(Px,Ty) + d(Sx,Qy)]/2\} < \varepsilon + h$$

$$\Rightarrow d(Px,Qy) < \varepsilon \qquad \text{.......(1)}$$

$Px = Qy$ whenever $Px = Sx$ and $Qy = Ty$. ....(2)

If the sequences of mappings $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively then the sequence $\{u_n\}$ converges to u uniquely.

Proof From (1), for all x, y in X for which $Px, \neq Sx$ and $Oy \neq Ty$, we have

$$d(Px,Qy) < max\ \{d(Sx,Ty),\ d(Px,Sx),\ d(Qy,Ty),$$
$$[d(Px,Ty) + d(Sx,Qy)]/2\} \qquad ....(3)$$

The point u_n is the common fixed point of P_n, Q_n, S_n and T_n fo reach $n = 1,2,\ldots$, so we have

$$P_n u_n = S_n u_n = u_n = Q_n u_n = T_n u_n.$$

Also, $Pu = Su = u = Qu = Tu$.

To establish the theorem, we first show that

$$lim_n\ d(Pu_n,Qu) = 0.$$

If $\lim_n d(Pu_n,Qu) \neq 0$, then there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$, a positive integer N and a number $r > 0$ such that

$$n_i \geq N \Rightarrow inf\ \{d(P_{u_{n_i}}, Qu)\} = r.$$

Further, since the sequenes $\{P_n\}$, $\{Q_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to P, Q, S and T respectively, given $h = h(r) > 0$ and for each $u_n \in X$ there exists a positive integer $M = M(h(r))$ such that

$$n \geq M \Rightarrow d(P_n u_n,\ Pu_n) < h/4,\ d(Q_n u_n, Qu_n) < h/4,$$
$$d\ (S_n u_n,\ Su_n) < h/4,\ d(T_n u_n,\ Tu_n) < h/4 \qquad ....(4)$$

Therefore, for $n \geq N$ we obtain

$$d(Pu_n, Qu) < max\{d(Su_n, Tu),\ d(Pu_n, Su_n),\ d(Qu, Tu),\ [d(Pu_n, Tu) + d(Su_n, Qu)]/2\}$$

$$\leq d(S_n u_n, Su_n) + d(S_n u_n, Pu_n) + d(Pu_n, Tu)$$

$$= d(Pu_n, Qu) + d(S_n u_n, Su_n) + d(P_n u_n, Pu_n)$$

$$< d(Pu_n, Qu) + h/2. \qquad \text{.......(5)}$$

Now, if we increase N then r and $h(r)$ do not increase, that is, if we increase N then M need not increase. Let us therefore, select $N > M$ and denote the corresponding value of r by r_N, that is,

$$n_i \geq N => inf\ \{d(P_{u_{n_i}}, Qu)\} = r_N .$$

This means there exists at least one value m of n_i such that $m \geq N$ and $d(Pu_m, Qu)$ is arbitrarily close to r_N and

$$d(Pu_m, Qu) > r_N . \qquad \text{.......(6)}$$

This, in view of (5), implies

$$r_N < max\{d(Su_m, Tu), d(Pu_m, Su_m), d(Qu, Tu), [d(Pu_m, Tu) + d(Su_m, Qu)]/2\}$$

$$< r_N + h(r_N). \qquad \text{.......(7)}$$

However, in view of (1) and (4), inequality (7) yields

$$d(Pu_m, Qu) < r_N .$$

This contradicts (6). Hence $lim_n\ d\ (Pu_n, Qu) = 0$. Then

$$lim_n \quad d(u_n, u) = lim_n \ d(P_n u_n, Qu)$$

$$\leq lim_n \ d(P_n u_n, P_n n) + lim_n \ d(Pu_n, Qu) = 0$$

Therefore, the sequence $\{u_n\}$ converges to u. Proof of uniqueness of u follows easily.

Thus the theorem is established.

Remark 1 : On taking $S_n = T_n$ and $P_n = Q_n$ for every value of n, in the above theorem, we obtain a generalized version of Theorem 4 of Singh [14] as a special case of our theorem.

Remark 2: In the above theorem, if we take $P_n = Q_n$ and $S_n = T_n$ = the identity mapping, we obtain Theorem 6.12 of Singh [15] as a particular case of our theorem. We also obtain theorem due to Dube and Singh (1972), see Singh [15], as a particular case of our theorem.

Remark 3 : In theorem 1, if we select $P_n = Q_n$ and $S_n = T_n$ = the identity map, then we obtain an extended form of the Theorem 4.6.8 of [4, p. 151] which is due to Bose and Mukherjee [1].

Theorem 2 : Let $\{P_i\}$ and $\{Q_j\}$, $i,j = 0,1,2, \ldots$, be sequences of selfmappings and let S and T be selfmappings of a complete metric space (X,d) satisfying the conditions:

Given $\varepsilon > 0$, there exists $h(\varepsilon) > 0$, $h(\varepsilon)$ being a nondecreasing function of ε, such that for all x, y in X

$$\epsilon \leq max\{d(Sx,Ty), d(P_ix,Sx), d(Q_jy,Ty), [d(P_ix,T_y) + d(Q_jy,Sx)]/2\} < \epsilon + h$$

$$\Rightarrow d(P_ix, Q_jy) < \epsilon \qquad \ldots\ldots(1)$$

$$P_ix = Q_jy \text{ whenever } P_ix = Sx \text{ and } Q_jy = Ty. \qquad \ldots\ldots(2)$$

Let the range of one of the mappings $\{P_i\}$, say Po, be contained in the range of T and the range of one of the mappings $\{Q_j\}$, say Q_0, be contained in the range of S. Also let P_0 and S, and Q_0 and T be weakly comuting. If one of the mappings P_0, Q_0, S and T be continuous then the mappings $\{P_i\}$, $\{Q_j\}$, S and T have a unique common fixed point which is also the unique common fixed point of $\{P_i\}$ and S and of $\{Q_j\}$ and T for every value of i and j.

Proof Let us first consider the mappings P_o, Q_o, S *and* T. These four mappings satisfy all the conditions of Theorem A. Therefore, by virtue of Theorem A, the mappings P_o, Q_o, S and T have a unique common fixed point, say z. Further, z is the unique common fixed point of P_o and S and of Q_o and T. Thus

$$P_oz = Q_oz = z = Sz = Tz.$$

Let us now consider a mapping P_i for some arbitrarily chosen value of i. Then $z = P_iz$. Otherwise, from (1) we get

$$d(P_i z,\ z) = d(P_i z, Q_o Z)$$

$$< max\{d(Sz,Tz),\ d(P_i z,Sz),\ d(Q_o z,Tz),$$

$$[d(P_i z,Tz) + d(Q_o z,Sz)]/2\}$$

$$= d(z,P_i z),$$

a contradiction. Similarly it can be shown that $z = Q_j z$ for any value of j. If possible, suppose u is a second common fixed point of P_i and S. Then from (1) we obtain

$$d(u,z) = d(P_i u.Q_0 z)$$

$$< max\{d(Su,Tz),\ d(P_i u,Su),\ d(Q_o z,Tz),\ [d(P_i u,Tz) + d(Q_o z,Su)]/2\}$$

$$= d(u,z),$$

a contradiction. Hence z is the unique common fixed point of P_i and S for each value of i. Similarly z is the unique common fixed pont of Q_j and T for every value of j.

This establishes the theorem.

Remark 4 : In Theorem 2, if we let $P_i = Q_i$ for every value of i and $S = T =$ the identity mapping, we obtain a generalized version of a theorem due to Bose and Mukherjee [1].

Remark 5 : Letting $P_i = Q_i$ for each i and $S = T$ in Theorem 2, an extended form of Theorem 1 of Tewari and Singh [16] is obtained.

Remark 6 : By an obvious modification in the proof of Theorem 2, that is, by using the properties of compatibility in place of those of weak commutativity at relevant places, the theorem pertains to compatible maps and we obtain a multitude of fixed point theorems as special cases of our theorem. For example, if we take $P_i = P$ for each i, we can obtain the theorem due to Rhoades, Park and Moon [13] under relaxed conditions as a special case of our theorem. The theorem of Rhoades, Park and Moon assumes stronger condition on compatibility and the ranges of the mappings. The theorems due to Fisher [2], Hardy and Rogers [3], Jungck [5], Kannan [6], Meir and Keeler [7], Pant [8, p. 78], [9], [10], Park and Bae

[11], Park and Rhoades [12], Singh [14], Tewari and Singh [16], [17] and Theorem 4.5.8 of [4] can also be obtained as special cases.

Remark 7 : A weaker form of Theorem 2, under the stronger condition that each P_i weakly commutes with S and each Q_j weakly comutes with T, can be obtained as a special case of Theorem 2.15 of [8, p. 89).

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A FIXED POINT THEOREM FOR SELF MAPPINGS ON MENGER SPACES

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ABSTRACT

In this paper a fixed point theorem in the context of Menger Spaces is proved. The theorem is supported by an example. A fixed point theorem in complete metric spaces also follows.

AMS Subject Classification (1990) Primary 54H25.

Keywords : Menger Space, fixed point, t-norm.

INTRODUCTION

A Menger Space is defined as a triplet $(M,F,t,)$ where M is a set, $F:MxM \rightarrow \Delta^+$ (set of distribution functions) satisfies the following

(a) $F_{xy}(0) = 0$, for all $x,y \ \varepsilon \ M$
(b) $F_{xy}(s) = 1$, for all $s>0$ holds if and only if $x=y$
(c) $F_{xy} = F_{yx}$, for all x,y ε M
and (d) $F_{xy}(s_1 + s_2) \geq t(F_{xz}(s_1), F_{zy}(s_2))$, for all x,y,z $\varepsilon \ M$ and $s_1, s_2 \geq 0$

where t is a t-norm, that is a function $t:[0,1] x [0,1] \rightarrow [0,1]$ which satisfies
(i) $t(a,1) = a, t(0,0) = 0$
(ii) $t(a,b) = t(b,a)$
(iii) $t(c,d) \geq t(a,b)$, $c \geq a$, $d \geq b$
and(iv) $t(a,t(b,c)) = t(t(a,b),c)$ for all $a,b,c,d \ \varepsilon \ [0,1]$

Examples of t-norm are $t(a,b) = min(a,b), t(a,b) = max\{a+b-1,0\}$etc.

A Sequence $\{p_n\}$ is said to converge to p if given $\varepsilon > 0$, $\lambda > 0$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that $F_{pp_n}(\varepsilon) > 1 - \lambda$ whenever $n \geq N_{\varepsilon,\lambda}$.

A Sequence $\{p_n\}$ is a Cauchy sequence if given $\varepsilon > 0$, $\lambda > 0$ there exists

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a positive integer $N_{\varepsilon,\lambda}$ such that $F_{p_m p_n}(\varepsilon) > 1-\lambda$ whenever $m,n > N_{\varepsilon,\lambda}$.

A Menger Space is complete if every Cauchy sequence is convergent.

MAIN RESULTS

In this section we obtain a fixed point theorem for a self-mapping on a complete Menger space.

Lemma-1. $\{p_n\}$ is a sequence in a Menger Space $(M, F, t \equiv min)$and $\{s_n\}$ is a sequence of positive real numbers such that $\sum_{i=0}^{\infty} s_i < \infty$ If $\mathop{\pi}\limits_{i=0}^{\infty} F_{p_i p_i}(s_i) > 0$ then$\{P_n\}$ is a Cauchy sequence.

Proof. None of $F_{p_i p_{i+1}}(s_i) = 0$. So let, $F_{p_i p_{i+1}}(s_i) = 1-\lambda_i$ where $0 \le \lambda_\iota < 1$, i = 1, 2..... Then, $\mathop{\pi}\limits_{i=1}^{\infty} (1-\lambda_i) > 0$ this implies ([3]) $\sum_{i=1}^{\infty} \lambda_i < \infty$. Given $\varepsilon > 0$ and $\lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that

$$\sum_{i=N_{\varepsilon,\lambda}}^{\infty} s_i < \varepsilon \text{ and } \sum_{i=N_{\varepsilon,\lambda}}^{\infty} \lambda_i < \lambda$$

Let $m, n > N_{\varepsilon,\lambda}$. Further let $n > m$.

$$F_{p_m,p_n}(\varepsilon) \ge F_{p_m p_n}(s_m + \ldots + s_{n-1})$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_n}(s_{m+1} + \ldots + s_{n-1})\right)$$

$$\ge t\left(F_{p_m p_{m+1}}(s_m), t\left(F_{p_{m+1} p_{m+2}}(s_{m+1}), t\left(\begin{array}{l} F_{p_{m+2}, p_{m+3}}(s_{m+2})\ldots, \\ t\left(F_{p_{n-2}, p_{n-1}}(s_{n-2})\, F_{p_{n-1} p_n}(s_{n-1})\right) \end{array}\right)\right)\right)$$

By virtue of properties of t-norm it follows that

$$F_{p_m p_n}(\varepsilon) \ge min\left\{F_{p_m p_{m+1}}(s_m), F_{p_{m+1} p_{m+2}}(s_{m+1}) \ldots, F_{p_{n-1}, p_n}(s_{n-1})\right\}$$

$$\ge min\{1-\lambda_m, 1-\lambda_{m+1} \ldots, 1-\lambda_{n-1})\}$$

or $F_{p_m p_{m+1}}(\varepsilon) > 1-\lambda$ whenever $m,n > N_{\varepsilon,\lambda}$. Thus $\{p_n\}$ is a Cauchy sequence.

Theorem-1. *Let $(M,F,t \equiv min)$ be a complete menger space.* T:M→M satisfies the following

a) $$F_{T_x,T_y}(s_1+s_2) \ge F_{x,T_x}\left(\frac{s_1}{\alpha}\right) + F_{T_y,T_y^2}\left(\frac{S_2}{\beta}\right) - 1$$

where $0 < \alpha < 1$ and $\beta > 0$ are constants and $s_1,\ s_2 \ge 0$

and b) there exists $\{x_i) \subset M$ and a sequence of real numbers $\{s_i\}$ with $\sum_{i=1}^{\infty} s_i < 1$ such that

$F_{x_i,T_{x_i}}(s_1) > 1-s_1$

$F_{T_{x_2},T^2{x_2}}(s_2) > 1-s_2$

and in general $F_{T_{x_i}^{i-1},T_{x_i}^{i}}(s_i) > 1-s_i,\ i=1,2\ldots$

Then, T has a unique fixed point.

Proof. $$F_{T^i_{x_i},T^i_{x_{i+1}}}(s_i+\beta s_{i+1}) \ge F_{T^{i-1}_{x_i},T^i_{x_i}}\left(\frac{s_i}{\alpha}\right) + F_{T^i_{x_{i+1}},T^{i+1}_{x_{i+1}}}\left(\frac{\beta s_{i+1}}{\beta}\right) - 1$$

Since, $0 < \alpha < 1, F_{T^i_{x_i},T^i_{x_{i+1}}}(s_i+\beta s_{i+1}) \ge F_{T^{i-1}_{x_i},T^i_{x_i}}(s_i) + F_{T^i_{x_{i+1}},T^{i+1}_{x_{i+1}}}(s_{i+1}) - 1$

$> 1 - s_i + 1 - s_{i+1} - 1$

$= 1 - s_i - s_{i+1}$

Thus we have,

$F_{x_i,T_{x_i}}(s_1) > 1-s_1$

$F_{T_{x_1},T_{x_2}}(s_1+\beta s_2) > 1-(s_1+s_2)$

$$F_{T_{x_2}, T^2_{x_2}}(s_2) > 1 - s_2,$$

$$F_{T^2_{x_2}, T^2_{x_3}}(s_2 + \beta s_3) > 1 - (s_2 + s_3),$$

$$F_{T^2_{x_3}, T^3_{x_3}}(s_3) > 1 - s_3$$

$$F_{T^3_{x_3}, T^3_{x_4}}(s_3 + \beta s_4) > 1 - (s_3 + s_4 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{etc.}$$

Also $0 < s_1, (s_1+s_2), s_2, (s_2+s_3) \dots\dots\dots < 1$ because $\sum_{i=1}^{\infty} s_i <$

Now the series, $s_1 + (s_1+s_2) + s_2 + (s_2+s_3) + (s_3+s_4) + \dots$ to
$= 2s_1 + 3s_2 + 3s_3 + \dots\dots\dots\dots\dots\dots$ to is convergent

So the infinite product,

$(1-s_1)(1-s_1-s_2)(1-s_2)(1-s_2-s_3) \dots\dots to\ \infty > 0$ ([3])

Hence, $F_{x_1, T_{x_1}}(s_1)\, F_{T_{x_1}, T_{x_2}}(s_1 + \beta s_2) . F_{T_{x_2}, T^2_{x_2}}(s_2)$
$. F_{T^2_{x_2}, T^2_{x_3}}(s_2 + \beta s_3) \dots to\ \infty > 0$

Also the series,

$$s_1 + (s_1 + \beta s_2) + s_2 + (s_2 + \beta s_3) + s_3 + (s_3 + \beta s_4) + s_4 + \dots \ to\ \infty.$$

$$= 2s_1 + (2 + \beta)\ s_2 + (2 + \beta)\ s_3 + \dots\dots \text{to } \infty$$

is convergent, since $\sum_{i=1}^{\infty} s_i <$

Therefore by Lemma-1 the sequence

$$\{\ x_1, T_{x_1}, T_{x_2}, T^2_{x_2}, T^2_{x_3}, T^3_{x_3}, T^3_{x_4}, T^4_{x_4} \dots T^{i-1}_{x_i}, T^i_{x_i}, \dots\ \}$$

is convergent. Let the sequence converge to x.

Let ε > 0 be arbitrary.

We choose α_1 such that $\alpha < \alpha_1 < 1$,

then $F_{x,T_x}(\varepsilon) \geq t(F_{x,T_{x_i}^{i-1}}((1-\alpha_1)\varepsilon)), F_{T_{x_i}^{i-1},T_x}(\alpha_1\varepsilon))$

$= t(F_{x,T_{x_i}^{i-1}}((1-\alpha_1)\varepsilon), F_{T_{x_i}^{i-1},T_x}(\alpha_2 + (\alpha_1 - \alpha_2)\varepsilon)$

(where we choose α_2 such that $\alpha < \alpha_2 < \alpha_2$)

$$\geq t(F_{x,T_{x_i}^{i-1}}((1-\alpha_1)\varepsilon), F_{x,T_x}(\frac{\alpha_2}{\alpha}\varepsilon) + F_{T_{x_i}^{i-1},T_{x_i}^{i}}(\frac{\alpha_1 - \alpha_2}{\beta}\varepsilon)) - 1)$$

Let n be an arbitrary positive integer. Since the sequence $\{x_1, T_{x_1}, T_{x_2}, T_{x_2}^2, T_{x_3}^2, T_{x_3}^3 \ldots\}$

converges to the point x we can choose i in such a way that

$F_{T_{x_i}^{i-1},T_{x_i}^{i}}(\frac{\alpha_1 - \alpha_2}{\beta}\varepsilon) > 1 - \frac{1}{n}$

and $F_{x,T_{x_i}^{i-1}}((1-\alpha_1)\varepsilon) > 1 - \frac{1}{n}$,

Therefore $F_{x,T_x}(\varepsilon) \geq t(1 - \frac{1}{n}, F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon) + 1 - \frac{1}{n} - 1)$

$\geq t(1 - \frac{1}{n}, F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}$

$\geq t(F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}, F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n})$

$F_{x,T_x}(\varepsilon) \geq F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon) - \frac{1}{n}$

Since n is arbitrary

$F_{x,T_x}(\varepsilon) \geq F_{x,T_x}(\frac{a_2}{\alpha}\varepsilon)$

Since $\frac{a_2}{\alpha} >$ we have $F_{x,T_x}(\varepsilon) = 1$ for all $\varepsilon > 0$.

That is $x = T_x$ is a fixed point. If possible, let $y = T_y$ be another fixed point of T.

Then from a) for $s > 0$

$$F_{T_x,T_y}(s) \geq F_{x,T_x}\left(\frac{s}{2\alpha}\right) + F_{T_y,T_y^2}\left(\frac{s}{2\beta}\right) - 1$$

or $$F_{x,y}(s) \geq F_{x,x}\left(\frac{s}{2\alpha}\right) + F_{y,y}\left(\frac{s}{2\beta}\right) - 1$$

or $F_{x,y}(s) \geq 1$ for $s > 0$.

Hence $x = y$
Therefore the fixed point is unique.

Cor-1. If (X,d) is a complete metric space and $T : \rightarrow X$ satisfies
1. $d(T_x, T_y) < \alpha d(x, T_x) + \beta d(T_y, T^2_y)$, where $0 < \alpha < 1$ and $\beta > 0$
11. T is onto
111. $\inf d(x, T_x) = 0$

Then T has unique fixed point

The proof of the Corollary will follow from the fact that a complete metric space can also be treated as a complete menger space if we put $F_{pq}(t) = H(t - d(p, q))$ [4], [6]. The inequality (1) in corollary implies the inequality (a) in the Theorem and conditions (11) and (111) in the corollary imply condition (b) of the Theorem. The proof then follows by the application of Theorem 1.

It may noted that fixed points on Menger space have been discussed in many works. We have noted some of them in references [1], [2] [5], [6].

Examples

We set an example to show the applicability of Theorem-1.
Let G any distribution function with $G(0)$ abd $0 < G(x) < 1$ for all $x > 0$. We consider the triplet (M,F,t) where $M = [0, \infty]$ t = min and F is defined as

$F_{pq} = H$, if $p = q$

where $H(x) = 1, x \,.\, 0$
$\quad\quad = 0, x < 0$

and $F_{pq}(s) = G(s / |p - q|)$
It is easily seen that (M,F,t) is a complete Menger space.

Let $T : X \to X$ be defined as $T_x = x/4$. Then

$$\left|T_x - T_y\right| = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}\ |x - y| \le \frac{1}{4}(|x| + |y|)$$

or, $\left|T_x - T_y\right| \le \frac{1}{4}.\ 2\ max\ \{|x|\ ,\ |y|\}.$

Therefore either Case I : $\left|T_x - T_y\right| \le \frac{1}{2}\ |x|$

or Case II : $\left|T_x - T_y\right| \le \frac{1}{2}\ |y|$

Case I : $\left|T_x - T_y\right| \le \frac{1}{2}\ |x| = \frac{1}{2}\frac{4}{3}\left|\frac{3}{4}x\right|$

$$\frac{2}{3}\left|x - \frac{1}{4}x\right| = \frac{2}{3}\left|x - T_x\right|$$

$$G\left(\frac{s}{\left|T_x - T_y\right|}\right) \ge G\left(\frac{s}{\frac{2}{3}\left|x - T_x\right|}\right),\ s > 0$$

$${}^F T_x, T_y(s_1 + s_2) \ge F_x, T_x\left(\frac{s_1}{2/3}\right) + {}^F T_y,\ T_y^2\left(\frac{s_2}{8/3}\right)$$

Case II : $\left|T_x T_y\right| \le \frac{1}{2}|y| = \frac{1}{2}\frac{16}{3}\left|\frac{3}{16}y\right|$

$$= \frac{8}{3}\left|y/4 - y/16\right| = \frac{8}{3}\left|T_y - T_y^2\right|$$

so $G(\frac{s}{\left|T_x - T_y\right|}) \ge G(\frac{s}{8/3\left|T_y - T_y^2\right|})s > 0$

or $F_{T_x T_y}(s) \ge F_{T_y T_y^2}(\frac{s}{8/3})$

Both cases imply inequality

$$F_{T_x, T_y}(s_1 + s_2) \ge F_x.T_x\left(\frac{s_1}{2/3}\right) + F_{T_y,\ T_y^2}\left(\frac{s_2}{8/3}\right)$$

$$\geq F_{x\,,T_x}\left(\frac{s_1}{2/3}\right) + F_{T_y\,,\,T_y^2}\left(\frac{s_2}{8/3}\right) - 1$$

where $s_1, s_2 > 0$.

Let $\{s_1\}$ be a sequence such that $\sum_{i=1}^{\infty} s_i <$. we choose $\{x_i\}$ such that

$$G(s_i / \left| x_i(\frac{1}{4^i} - \frac{1}{4^{i-1}}) \right|) > 1 - s_i,\ i = 1,2$$

so that $F_{T^{i-1}_{x_i}, T^{i}_{x_i}}(s_i) > 1 - s_i,\ i = 1,2\ldots$

Hence b) of Theorem-1 is satisfied with the choice of sequence $\{x_i\}$ and $\{s_i\}$

Thus by application of Theorem-I it follows that T has a unique fixed point. It may be seen that `$O$` is the unique fixed point.

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FIXED POINTS IN FUZZY METRIC SPACES

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ABSTRACT

In this note we establish Banach type fixed point theorems on fuzzy metric spaces (FM-spaces). Our results generalize several known fixed point theorems on FM-spaces and extend certain results of Hicks-Rhoades and others.

Mathematics Subject Classifications (1991) : 54H25, 54A40, 54E99.

Keyword and Phrases : Fuzzy metric space, fixed point, orbitally upper semicontinuous.

INTRODUCTION

Following Kramosil-Micha'lek's [5] approach to fuzzy metric spaces (FM-spaces), Grabiec [2] has formulated fixed point theorems for contractive maps on FM-spaces. Subsequently, Fang [1], Mishra et al. [6] and Singh et al. [7] have investigated existence of solutions of abstract fixed point equations on FM-spaces. The intent of this note is to establish fixed point theorems for Banach type maps on FM-spaces. Our results extend and fuzzify fixed point theorems of Hicks-Rhoades [3], Hicks [4] and others.

PRELIMINARIES

We shall generally follow the notations and definitions of Grabiec [2], Hicks-Rhoades [3], and Mishra et al. [6].

A fuzzy metric space is an ordered triplet $(X, M, *)$ consisting of a nonempety set X, a fuzzy set M in $X^2 \times (0, \infty)$ and *, a continuous t-norm [2] (see, also [5], [6]). The functions $M(x, y, .) : [0, \infty) \to [0, 1]$ are left continuous and are assumed to satisfy the following conditions :

(FM-1) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$
(FM-2) $M(x, y, 0) = 0$
(FM-3) $M(x, y, t) = M(y, x, t)$

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(*FM-4*) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$, *for all* $x, y, z \in X$ and $t, s > o$.

Grabiec [2] has shown that $M(x, Y, *)$ is nondecreasing for all $x, y \in X$. For details of topological preliminaries, refer to [2] and [5].

A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim M(x_n, x, t) = 1$ for each $t > o$, and $\{x_n\}$ is a Cauchy sequence if $\lim M(x_{n+p}, x_n, t) = 1$ for each $t > 0$, and $p > 0$. An FM-space, in which every Cauchy sequence is convergent, is called complete.

In all that follows, X stands for an FM-space $(X, M, *)$ with the following property :

(*FM-5*) $\lim_{t \to \infty} M(x, y, t) = 1$ *for all* $x, y \in X$.

FIXED POINT THEOREMS

Defination 1. Let X be an FM-Space and $T : X \to X$. Orbit of T at x, denoted by $O(x, \infty)$ is defined as $O(x, \infty) = \{x, Tx, T^2x.....\}$.

Defination 2. Let X be an FM-space. A function G from X to $[0, 1]$ is *T-orbitally upper semicontinuous at* $z \in X$ if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $\{x_n\}$ converges to z implies $G(z) \geq \lim\sup G(x_n)$.

The following result is essentially due to Grabiec [2].

Theorem 1. Let X be an FM- space. If $T: X \to X$ satisfies

(1.1) $M(Tx, Ty, kt) \geq M(x, y, t)$;

for all $x, y \in X$, *and* $o < k < 1$, *then T has a unique fixed point.*

The above result is known as *fuzzy Banach contraction principle.*

Theorem 2. *Let X be an FM-space and* $T : X \to X$. *If X is T-orbitally complete and for an* $x \in X$,

(2 .1) $M(Ty, T^2 y, kt) \geq M(y, Ty, t)$;

$y \in O(x, \infty)$, $k \in (0, 1)$, *then for an* $x_o \in X$, *there exists an orbit* $O(x_o)$ *converging to a point* $z \in X$. *Further :*

(*a*) *z is a fixed point of T if T is orbitally continuous.*

(*b*) *z is a fixed point of T if and only if $G(x) = M(x, Tx, t)$ is T-orbitally upper semicontinuous at z.*

Proof. Pick $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1} = T^n x_0$
In view of (2.1),

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, T^2x_{n-1}, t) \\ &\geq M(x_{n-1}, Tx_{n-1}, t/k) \\ &\geq \ldots \geq M(x_0, x_1, t/k^n). \end{aligned}$$

This shows that $\{x_n\}$ is a cauchy sequence in X, hence convergent. Call the limit z.

(a) Since
$M(Tz, z, t) \geq M(Tz, Tx_n, t/2) * M(x_{n+1}, z, t/2)$

and T is continuous, making $n \to \infty$ yields $M(Tz, z, t) \geq 1*1 = 1$

Thus z is a fixed point of T.

(b) Let $z = Tz$ i .e $M(z, Tz, t) = 1$ Then $G(z) = M(z, Tz, t) = 1 \geq \lim\sup G(x_n)$, implying $G(x)$ is T-orbitally upper upper semicontinuous at z.

Conversely, if $G(x)$ is T-orbitally upper semicontinuous at z, then $G(x_n) = M(x_n, Tx_n, t) \leq M(z, Tz, t)$, yielding
$M(z, Tz, t) = 1$, *i.e.*, $z = Tz$.

Following theorem shows that Theorem 2 is more general than Theorem 1.

Theorem 3. *Let X be an FM-space and $T: X \to X$. Then the condition (1.1) implies the condition (2.1)*

Proof. Taking $y = Tx$ in (1.1), we get the condition (2.1).

Here we remark that our result is superior to several results on the existence of fixed points in FM-spaces. But, first, consider the following for $T:X \to X$. For all $x, y \in X$,

(3.1) $$M(Tx, Ty, kt) \geq M(x, y, t) * M(x, Tx, t) * M(y; Ty, t) * M(x, Ty, \alpha t) * M(y, Tx, (2-\alpha)t).$$

where $k \in (0, 1)$, $t > 0$ and $\alpha \in (0, 2)$.

Recently Mishra et al. [6] and Singh et al. [7] studied the condition (3.1) and obtained some interesting results on FM-spaces. Following comparing theorem shows that our result also generalize several fixed point theorems for above contractive condition.

Theorem 4. Let X be an FM-space and $T : X \to X$. Then the condition (3.1) implies the condition (2.1).

Proof. Taking $y=Tx$ in (3.1). We get for $\alpha = 1+q$, $q \in (0, 1)$

$$\begin{aligned} M(Tx, T^2x, kt) &\geq M(x, Tx, t)*M(Tx, T^2x, t)*M(x, T^2x, (1+q)t)*M(Tx, Tx, (1-q)t) \\ &\geq M(x, Tx, t)*M(x, Tx, qt). \end{aligned}$$

Since the norm * is continuous and $M(x, y, .)$ is left continuous, making $q \to 1$ gives $M(Tx, T^2x, kt) \geq M(x, Tx, t)$. This completes the proof.

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ON SOME PROPERTIES OF DISTANCE SETS

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ABSTRACT

Steinhaus proved that the set of differences A-B contains an interval whenever A and B are linear sets of positive Lebesgue measure. S.Piccard proved that the same conclusion holds when A and B have the property of Baire. In the present note we strengthen the results of Steinhaus as well as Piccard.

Mathematics subject classification (1991): 28A 05

Key words and Phrases : Distance set, Property of Baire, Residual set.

INTRODUCTION

Let A and B be subsets of the real line R. Then the distance set of A and B is defined to be the set $D(A,B) = \{|x-y| : x \in A, y \in B\}$. The investigation of the set of distances between the points of a measurable set was started by H. Steinhaus. In 1920, Steinhaus [7] proved that if $A, B \subset R$ be sets with positive Lebesgue measure then $D(A,B)$ contains an interval. A linear set is said to have the "Property of Baire" if it can be expressed as the symmetric difference of an open set and a set of first category. A set of the second category which is complement to a set of the first category is called a residual set [4]. Therefore, there are two kinds of sets of the second category, those which are complement to a set of the first category and those for which this is not the case. It is also known that the intersection of a sequence of residual sets is also residual. In 1939, S. Piccard [5] proved the result same as that of Steinhaus when A and B were two linear sets of second category having the property of Baire. Many papers are devoted to the study of the distance set $D(A)$ for various A. The corresponding problem for sets in n-dimensional space was considered by Kestelman [3] in 1947, and he proved a number of elegant theorems about distance sets in n-dimensional space and Steinhaus theorem came out as a particular case of one of his results. The question has been asked if there is

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a set of measure zero for which the distance set contains an interval with origin as a left hand end point. In 1917, Steinhaus [6] proved that the Contor set C is an example of a set of measure zero whose distance set contains an interval with origin as a left hand end point. Bose Majumdar [2] made an exhausted study of $D(C)$. Boas [1] has proved that for almost all d in $[0,1]$ there is an uncountable (of the power of the continuum) set of pairs $(x,y) \in C \times C$ such that $|x-y| = d$. In connection with this, the question arises what is the situation for the sets of positive measure.

Steinhaus [7] proved the following interesting result: *Let A and B be linear sets with positive measure and C be a set of real numbers everywhere dense; then there exist two points 'a' and 'b' belonging to A and B respectively such that the number* $|a-b|$ *belongs to C*. As a consequence of this result he proved that $D(A,B)$ for sets of positive measure A and B, contains an interval.

In this paper we have attempted to generalise this result. For some d in a countable dense subset of R, the equality $|x-y| = d$ holds for an uncountable (of the power of the continuum) number of pairs $(x,y) \in A \times B$. We obtain the same result as above when A and B are taken as sets of second category having the property of Baire.

Theorem 1. *Let A and B be two linear sets of positive measure and C be a countably dense subset of R. Then for some* $d \in C$, *the set* $E = \{x \in A : \exists\, y \in B$ *with* $|x-y| = d\}$ *has positive measure.*

Proof. If possible let E be of measure zero. Then for each $d \in C$, the set $\{x \in A : \exists y \in B$ such that $|x-y| = d\}$ has measure zero. Therefore the set $F = \{x \in A : \exists y \in B$ such that $|x-y| = d,\ d \in C\}$ also has measure zero.

Let $H = A \setminus F$. Then $\lambda(H) = \lambda(A) > 0$, where λ denotes the Lebesgue measure. Then according to Steinhaus [7] the set $B \setminus H$ contains an interval. Hence $C \cap (B \setminus H) \neq \Phi$. Then for some $y \in B,\ c \in C$ and $x \in H \subseteq A$ we have $y = x + c$. Thus $x \in F$-a contradiction. Hence the result.

Note. For some d in a countably dense subset C of R, the equality $|x-y| = d$ holds for an uncountable (of the power of continuum) number of pairs $(x,y) \in A \times B$.

Steinhaus [7] proved that $D(A,B)$ contains an interval where A and B are two linear sets of positive measure. Let us define the set $D_1(A,B)$ consisting of all those points d which can be expressed into the form $d=|x-y|$ for infinite number of pairs of points $x\in A$, $y\in B$. Obviously, $D_1(A.B)\subset D(A.B)$.

We have established the following result which is an easy consequence of Steinhaus's result :-

Theorem 2. If A and B are two linear sets of positive measure, then $D_1 \equiv D_1(A,B)$ contains at least one interval.

Proof. Suppose not, then D_1 does not contain an interval. So $R\backslash D_1$ is a dense subset of R. Then, we may choose C to be countably dense subset of $R\backslash D_1$. By the note after theorem 1 there exists $d\in C$ such that d is the distance between infinite number of points of A and B. Hence $d\in D_1$--*a* contradiction.

Therefore, D_1 contains at least one whole interval.

Theorem 3. *Let A and B be two linear sets of second category having the property of Baire. If C is a countable dense subset of R, then for some* $d\in C$, *the set* $\{x\in A: x\pm d\in B\}$ *is of second category.*

Proof. If not, then for every $d\in C$, the set $F_d=\{x\in A: x\pm d\in B\}$ is of first category.

Then the set $\bigcup_{d\in C} F_d$ is also of first category in R.

Therfore $E = A - \bigcup_{d\in C} F_d$ is a residual set in R.

Then according to Piccard [5], $B\backslash E$ contains an interval. Hence, $C\cap(B\backslash E)\neq\Phi$. Then for some $d\in C$, $b\in B$ and $a\in E\subseteq A$ we have $b=a+d$ and hence $a\in F_d$--*a* contradiction.

Hence the result.

Theorem 4. *Let C be a countable dense subset of R and I be a set of first category. Then there exists a residual set P such that,* $\forall x\in P$, $x+c\notin I$ *for any* $c\in C$.

Proof. Let us enumerate C as $e_1, e_2, \ldots\ldots e_n\ldots\ldots$ Let $R_n=\{r\in R: e_n+r\notin I\}$ $(n=1,2,\ldots)$. Then $R_n=[R-I](-e_n)$, which is a residual set for every n. *Let*

$P = \bigcap_{n=1}^{\infty} R_n$ Then P is a residual set and $\forall x \in P,\ x+e_n \notin I$ for any n.

Hence the proof.

Theorem 5. *If A and B are two residual linear sets in some open interval, then the set of all points d for which* $\{x \in A : x+d \in B\}$ *is of second category, contains an interval.*

Proof. Let E be the set of all those points d for which the set $\{x \in A : x+d \in B\}$ is of second category.

We first show that E hits every countable dense subset C of R.

If not, then for each $d \in C$, the set $\{x \in A : x+d \in B\}$ is of first category. Then set $F = \{x \in A : x+d \in B,\ d \in C\}$ is also of first category. Then $G = R \setminus F$ is a residual set having the property of Baire. So according to Piccard [5], $B \setminus G$ contains an interval. Then $(B \setminus G) \cap C \neq \Phi$, which gives rise to a contradiction.

Now, if E contains no interval, then there exists a countable dense subset C of R such that $E \cap C = \Phi$-- which is again a contradiction.

Hence the theorem.

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NONLINEAR HYBRID CONTRACTIONS

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ABSTRACT

Standard fixed point theorems state conditions under which there exist solutions to $Tx = x$ for the single-valued self-map T of a space X or to $y \in Py$ for a multivalued map P on X with values in its power set. In this paper, inter alia, we present a brief historical account of coincidence/fixed point theorems for contracting single- and multi-valued maps.

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Keywords and Phrases Coincidence/fixed point, multivalued map, weakly commuting maps, compatible maps.

INTRODUCTION

The well-known Brouwer fixed point theorem (*viz. a continuous map on a closed unit ball of* R^n *has a fixed point*) has maintained its magnificient presence in mathematics for the last 80 years, excluding, of course, a gestation of 2-3 years. This "fixed point"magnet may be called the queen of fixed point theory and, perhaps, of macro-analysis and its applications. It has many beautiful children, grand and grand-grand children such as, with the risk of missing many champion theorems, Birkhoff-Kellog (1922). Lefschetz (1926). Schauder (1927). Tychonoff (1935). Kakutani (1941), Hukuhara (1950), Glicksberg (1952), Darbo (1955), Browder (1959, 67), Jones (1963), Sadovskii (1967), Ky Fan (1961, 66, 69, 84). Reih (1972, 78, 79), Lin (1979), V.M. Sehgal & S.P. Singh (1985) and Bridges et al. (1992, cf. [5]). One may refer [41]) and [52] for the recent development of these theorems. However, beauty (of these results) has to pay the price of

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cosmetics in kind of "compactness in some sense. " " - - - the spaces which have the fixed point property for continuous copact set-valued mappings constitute a fairly small subclass of those which has the fixed point proerty for continuous single-valued mappings; (see Nadler [33, p. 476]). " This, perhaps, inspires the development of fixed point theory for contracting maps. In this paper, we are concerned about the study of fixed points of contracting multivalued maps "majorized" by single-valued maps, which started about a decade ago. A *prima facie* love for undertaking such a study appears to be H.W. Corley's formulation [7, Th. 1] of an abstract maximization problem in terms of a (hybrid) stationary point theorem.

PRELIMINARIES

We will use the following notation where (X, d) is a metric space :

$CL(X) = \{A \subset X: A \text{ is nonempty and closed}\}$,

$CB(X) = \{A \subset X: A \text{ is nonempty, closed and bounded}\}$,

For nonempty subsets A, B of X and $s > 0$,

$N(s, A) = \{x \in X : d(x, a) < s \text{ for some } a \in A\}$,

$E_{A,B} = \{s>0 : A \subset N(s,B),\ B \subset N(s,A))\}$,

$$H(A,B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi \\ +\infty & \text{if } E_{A,B} = \phi, \end{cases}$$

and for any $x \in A$, $d(x, A)$ denotes the distance between x and A. H is called the *Hausdorff* (resp. *generalized Hausdorff*) distance function for $CB(X)$ (resp. $CL(X)$) induced by d. In all that follows *id* stands for the identity map on X.

Definition 1 [34], [48]

Maps $f: X \to X$ and $T: X \to CL(X)$ are weakly commuting at $z \in X$ if $H(fTz, Tfz) \leq d(fz, Tz)$. f amd T commute weakly on X if they commute weakly at every point of X.

For an equivalent formulation for two single-valued maps on a metric space, see [44].

Definition 2 [24]

Maps $f : X \to X$ and $T : X \to CB(X)$ are compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$ whenever x_n is a sequence in X such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$.

For an equivalent formulation for two single-valued maps on a metric space refer [19-20]. We remark that commuting maps are weakly commuting and weakly commuting maps are compatible, and the reverse implication is not true (see, for instance, [19-20], [24], [34], [44-45] and [47-48]). However, all the three concepts for a pair of maps are equivalent at a concidence point of the maps.

Let $T : X \to CL(X)$ and $f : X \to X$. We define (T, f) to be:

hybrid contraction if

(1) $H(Tx, Ty) \leq kd(fx, fy)$ for some $0<k<1$, and x,y in X;

hybrid nonexpansive if

(2) $H(Tx, Ty) \leq d(fx, fy)$, x,y in X; and

hybrid contractive if

(3) $H(Tx, Ty) < d(fx, fy)$, $fx \quad fy$ and x,y in X.

These definitions are modeled respectively on Banach contraction, nonexpansive and Edelstein's contractive maps [9-10] when $T : X \to X$ and $f = id$. Maps satisfying variants of (1)-(3) will be called hybrid type contraction/nonexpansive/contractive. For example, T and f satisfing.

(4) $H(Tx, Ty) \leq k \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), (1/2)[d(fx, Ty) + d(fy, Tx)]\}, 0<k<1$ and $x,y \in X$,

will be called '**hybrid type contraction**'. If no distinction is needed in a particular situation, maps falling in either of the three or similar categories will generally be described as **hybrid contracting maps** (Note that (1) implies (i), $i = 2,3,4$.)

The study of hybrid contracting maps was initiated during 1981-83 by Bhaskaran-Subrahmanyam [3] (see also [43], p.3), Hadzic [12], Kaneko [21], Mukherjee [32], Naimpally et al. [35] and Singh-Kulshrestha [49]. Indeed, the following coincidene theorem has a constructive proof in [49].

Theorem 1

Let $T: X \to CL(X)$ and $f: X \to X$ be such that (4) obtains, $f(X)$ is a complete subspace of X and

(1.1) $T(X) \subset f(X)$.

Then T and f have a coincidence. Indeed, for any x_o in X, there exists a sequence $\{x_n\}$ in X such that

(i) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots;$

(ii) $\{fx_n\}$ converges to fz for some z in X, and $fz \in Tz$,

i.e., T and f have a coincidence at z; and

(iii) $d(fx_n, fz) \leq (k^{1-q})^n (1-k^{1-q})^{-1} d(fx_o, fx_1)$,

where $q < 1$ is any positive number.

Coincidence theorems for (T,f) satisfying (1) and (1.1) appear in [12], [21] and [32] under strong conditions (such as continuity, commutativity) on the maps, Extensions of Theorem 1 appear in [42-43] and [48]. We remark that the condition (1.1) is sufficient for the existence of a sequene $\{fx_n\}$ but not necessary. Theorem 1 with $f = id$ is an important generalization of the multivalued contraction principle [8, 23] due to 'Ciric' [6, Th.2]. Indeed, the condition (4) of Theorem 1 includes several contraction type conditions for a multivalued map T on X with values either in $CB(X)$ or $CL(X)$ studied among others by Covitz-Nadler [8], Iseki [15], Kaneko [22-23], Kita [26, p. 115]. Nadler [33], Ray [39], Reich [40] and Wegrzyk [57]. Further, using 'Ciric's result [op.cit.] and following [35, Th. 2], one may also prove that T and f under the conditions of Theorem 1 have a coincidence.

Example 1

Under the conditions of Theorem 1, one can not expect, in general, a common fixed point for the maps T and f. In fact, let $X = [0, \infty)$, $Tx = [1+x, \infty)$ and $fx = 2x$ for x in X (see [35], [43]). Then all the hypotheses of Theorem 1 are satisfied, and T, f are common fixed point free. Indeed, T has no fixed point in X.

Example 2

Condition (1.1), although sufficient, is not necessary to obtain a coincidence. Let $X = [0, \infty)$, $Tx = [0, e^x]$ and $fx = 2(e^x - 1)$, then $f(X)$ is complete, (4) obtains, (1.1) fails and T and f have a coincidence, in fact, a common fixed point.

SOME HISTORICAL NOTES

The condition (1) with T single-valued on X was first studied by R. Machuca [29] in 1967 followed by Goebel [11] in 1968 (see (1*) below). In a symposium on Fixed Point Theory and its Applications (Meerut University, Meerut, 1982), papers of Bhaskaran-Subrahmanyam and Singh-Kulshrestha attempting to investigate conditions under which (T, f) satisfying (1)/(4) could have a common fixed point were a subject of discussion (see also [43], p. 3). Mukherjee [32] attempted, in vain, to establish a common fixed point theorem for continuous and commuting maps T and f satisfying (1) and (1.1) with X complete. Olga Hadzic [12] studied the following hybrid type contraction condition for $f, g : X \to X$ and $T : X \to CB(f(X) \cap g(X))$:

(5) $H(Tx, Ty) \leq kd(fx, gy),\ x, y \in X,$

and attempted a solution to $fx \in Tx$ and $gx \in Tx$, wherein f, g are continuous and T is closed. Hadzic's result is a special case of Theorem 4 (see below).

Let Y be an arbitrary set and $f, g: Y \to X$ such that for $0 < k < 1$ and x, y in Y,

(1*) $g(Y) \subset f(Y)$ and $d(gx, gy) \leq kd(fx, fy)$.

Machuca [op. cit.] attempted to show the existence of a coincidence point of f and g satisfying (1*) under heavy topological conditions. Goebel

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+ In a chat (Thunder Bay , 1983), Prof, G. Jungck denied the knowledge of [29]and [11].

[op. cit.] using the Banach contraction principle proved that the maps f and g have a coincidence when (1*) obtains and $f(Y)$ is complete. In 1976, Jungck [17], perhaps unaware⁺ of Machuca [op. cit.] and Goebel [op. cit.], gave a constructive proof to show that continuous commuting maps f and g satisfying (1*) with $Y=X$ complete have a unique common fixed point. This fixed point theorem together with [18] led to a massive growth of fixed point theorems for commuting, weakly commuting and compatible maps; we cite [2-4], [12-13], [19-21], [24], [28], [31-32], [34-35], [37-38] and [42-50] as examples.

A metric space X is metrically convex if for any distinct x, y in X, there exists z in X distinct from x,y such that $d(x,z) + d(z,y) = d(x,y)$. Let X be a complete and metrically convex metric space, and K a nonempty closed subset of X. Let $f\colon K \to X$ be a continuous map and $S, T\colon K \to CB(X)$ such that for all x,y in K,

$$\text{(BS)}\ H(Sx, Ty) \leq \alpha\, d(fx, fy) + \beta\, [d(fx,Sx) + d(fy, Ty)]$$
$$+\ \gamma\, [d(fx, Ty) + d(fy, Sx)]$$

Where $\alpha, \beta, \gamma \geq 0$ with $(\alpha + \beta + \gamma)(1 + \alpha + \gamma)/(1-\beta-\gamma)^2 < 1$. Assad-Kirk [1] studied (BS) with $S = T$, $f = id$ and $\beta = \gamma = 0$ by levying certain boundary conditions on the map T. Subsequently, Itoh [16], Khan [25], Tan-Minh [55], Yanagi [58] and others generalized their result [1, Th.1] for multivalued maps on K with values in $CB(X)$. Initial coincidence/fixed point theorems under (BS) and (BS) with $S = T$ obtained in [4] extend certain nice fixed point theorems of Itoh [op.cit.] and Khan [op. cit.].

Following Assad-Kirk [1], Bhaskaran-Subrahamanyam [4] (see also [3]) initiated the study of hybrid nonexpansive maps T and f on a subset of a banach space, and obtained coincidence/fixed point theorems for such maps. On the other hand, Hadzic' [13] initiated the study of hybrid nonexpansive maps and its variants on convex metric spaces (convexity in the sense of Takahashi, see [13]) and convex probabilistic metric spaces.

COINCIDENCE AND FIXED POINT THEOREMS FOR HYBRID TYPE CONTRACTIONS

Several generalizations, variants and special cases of Theorem 1 have been obtained on several settings (see, for instance, [21], [24], [28], [34].

[35], [37], [42], [43], [45], [48], [50] and [60-61]). Kaneko [21] proved that continuous and commuting maps $T{:}X \to CB(X)$ and $f{:}X \to X$ satisfying (1) and (1.1) have a coincidence in X provided X is complete. Generalizing this result Kaneko-Sessa [24] obtained the following :

Theorem 2

Let X be a complete metric space, and $f\colon X \to X$ and $T\colon X \to CB(X)$ be compatible continuous maps such that (1.1) and (4) hold. Then T and f have a coincidence.

We remark that the conditions of this theorem do not guarantee the existence of a common fixed point of T and f (see, for instance, Example 2 below). However, they [24, p. 260] (see also [21, Th. 1]) have shown that T and f with all the assumptions of Theorem 2 have a common fixed point whenever one of the following holds : either $fx \neq f^2x$ implies $fx \notin Tx$, or $fx \in Tx$ implies that $f^nx \to z$ for some $z \in X$. These conditions appear to be more acceptable than those cosmetized in [43].

In [50], Singh-Pant have obtained a coincidence theorem for two multivalued maps and a single-valued map in probabilistic analysis. However, the metric analogue of their result is a slightly improved by Theorem 3 (below), which is a special case of the main result of Singh-Ha-Cho [48].

Theorem 3

Let X be a metric space, $S,T{:}X \to CL(X)$ nd $f\colon X \to X$ such that $S(X) \cup T(X) \subset f(X)$. $f(X)$ is a complete subspace of X and

$$(6)\quad H(Sx, Ty) \leq k \max \{d(fx,fy),\, d(fx,Sx),\, d(fy,Ty),\, (1/2)[d(fx,Ty)+d(fy,Sx)]\}$$

for some $0<k<1$ and all x,y in X. then S, T and f have a coincidence, i.e., there exists a point z in X such that $fz \in Sz \cap Tz$. Further, if for such a z, fz is a fixed point of f, then fz is also a fixed point of S (resp. T) provided f and S (resp. T) are weakly commuting at z, fz is a common fixed point of S and T provided f weakly commutes with each of S and T.

Indeed, in [48], the conclusions of Theorem 3 are obtained for conditions more general than those in Theorem 3. We remark that, as the following example shows, without the assumption "fz is a fixed point of f" (or without conditions leading to it) in Theorem 3, f, S and T need not have a common fixed point, even if the maps are continous and commuting on X. Moreover, "weakly" in Theorem 3 is superfluous, since, recall that, the weak commutativity at a coincidence point of T and f is equivalent to the commutativity at the coincidence point of T and f. Moreover, several results of [59] appear to be special cases of [48].

Example 2 [48]

Let $X = [0,1]$ and $Sx = Tx = \{0,1\}$, $fx = 1-x$ for all x in X. Then all the hypotheses of Theorem 3 are satisfied except that none of the coincidence values, viz., $f0$ or $f1$, is a fixed point of f. Note that f and S are continuous and commuting on X.

Coincidence and fixed point theorems for two multivalued and two single-valued maps on a metric space have been obtained among others by Kubiak [28], Naidu-Prasad [34], Naidu [61], Popa [37] and Sessa-Fisher [45]. Following Hadzic [12], Kubiak [28] established the following coincidence theorem.

Theorem 4 [28]

Let X be a complete metric space, f and g continuous maps from X into X, S and T closed maps from X into $CB(f(X) \cap g(X))$ such that $fSx = Sfx$, $gTx = Tgx$ for every x in X and

$$H(Sx, Ty) \le k \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), (1/2)[d(fx, Ty) + d(gy, Sx)]\} \tag{7}$$

for some $0<k<1$ and all x, y in X. Then there exists a point z in X such that $fz \in Sz$ and $gz \in Tz$.

Now, we have the following coincidence theorem for a sequence of multivalued maps, wherein N is the set of natural numbers.

Theorem 5

Let X be a complete metric space, f and g continuous maps from X into X and $\{Tn : n \in N\}$ a sequence of closed maps from X into $CL(f(X) \cap g(X))$ such that, for each n in N and each x is X, $fT_nx = T_nfx$, $gT_nx = T_ngx$, and

(8) $H(T_mx, T_nx) \le$

$k \max \{d(fx,gy), d(fx,T_m x), d(gy,T_n y), 1/2[d(fx,T_n y)+d(gy,T_m x)]\}$
for some $0<k<1$ and all x,y in X, $m \neq n$. Then there exists a point z in X such that $fz \in T_n z$ and $gz \in T_n z$ for each $n \in N$.

Proof

An appropriate blend of the proofs of Kita [26] and Kubaik [28] will yield the result.

Note that Theorem 5 with $T_{2n-1} = S$ and $T_{2n} = T$ for each n in N slightly improves Theorem 4. Following Pal-Maiti [36] and the proof of Theorem 1 (above) (see [49]) we have the following :

Theorem 6

Let X be a metric space, $f: X \to X$ and $T : X \to CL(X)$ such that (1.1) obtains and $f(X)$ is a complete subspace of X. Further, let T and f satisfy, for x,y in X, at least one* of the following :

$$H(fx,Tx)+H(fy,Ty) \leq ad(fx,fy),\ 1<a<2;$$

$$H(fx,Tx)+H(fy,Ty) \leq b\ [d(fx,Ty)+d(fy,Tx)+d(fx,fy)], 1/2<b<2/3;$$

$$H(fx,Tx)+H(fy,Ty)+H(Tx,Ty) \leq c\ [d(fx,Ty)+d(fy,Tx)], 1< c <3/2;$$

and the condition (4) with $0<k<1$. Then there exists a point z in X such that $fz \in Tz$. Further, if fz is a fixed point of f, and if f and T commute at z, then fz is also a fixed point of T.

We remark that, in Theorem 6, if $f = id$ on X, then we get a result from [51], and if, in addition, T is a single-valued map on X, then we get an important fixed point theorem of Pal-Maiti [36].

As regards hybrid type contractions on a product space, a coincidence theorem has recently been obtained in [2] for two systems of multivalued maps and a system of single-valued maps on a finite product of metric spaces. It has been done by using a fixed matrix of non-negative numbers

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*The intended meaning is that, for each pair of points x, y of X, one (or a set of more than one) of the four conditions is to be satisfied.

(see [30]) and a new class of coordinatewise weakly commuting systems of single and multi-valued maps on the product space. It is too complicated to recall those results here. For an equivalent formulation of coordinatewise weakly commuting two sets of single-valued maps, refer [47].

HYBRID CONTRACTIVE MAPS

The following appears in [35].

Theorem 7

Let X be a metric space, $T : X \to CL(X)$ and $f: X \to X$ be such that $T(X) \subset f(X)$ and $f(X)$ is a compact subset of X. Suppose that for every $x, y \in X$,

$$(9) \qquad H(Tx, Ty) < d(fx, fy), \text{ if } fx = fy$$
$$Tx = Ty \text{ if } fx = fy$$

Then T and f have a coincidence.

For an attempt ;to obtain the conclusioin of this theorem under the strict inequality condition (4) with $k=1$, see [38]. Theorem 7 includes the following variant of Naimpally et al. [35, Cor. 3] (see also [11, p. 734]).

Theorem 8

Let X be a metric space and $f, g : X \to X$ be such that $g(X) \subset f(X)$, $f(X)$ is compact and

$$(10) \qquad d(gx, gy) < d(fx, fy), fx \neq fy$$

for x, y in X. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 8 includes certain important fixed point theorems of Edelstein [9-10] and Jungck [18]. For more general results, refer [20]. If $X = R^n$ we have the following improved form of Theorem 8.

Theorem 9 [46]

Let $f, g : R^n \to R^n$ be such that $g(R^n) \subset f(R^n)$, g is bounded and

$$\| gx - gy \| < \| fx - fy \| \quad , fx \neq fy$$

for x, y in R^n. Then f and g have a coincidence. Further, if f and g commute at their coincidence points, then f and g have a unique common fixed point.

Theorem 9 with $f = id$ on R^n is an important result of Bridges et al. [5, Th. 2.2], since a contractive map of the closed unit ball in a Banach space need not have a fixed point (see [56] and also [53, p. 39]).

One may ask if Theorem 9 can be extended to the case when g is a multivalued map. In particular, we have the following :

Question : Let $T : R^n \to CL(R^n)$ and $f : R^n \to R^n$ such that $T(R^n) \subset f(R^n)$, $f(R^n)$ is bounded and (9) holds for $x, y \in R^n$. Will T and f have a coincidence ?

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APPLICATION OF MODERN ASTRONOMY TO CERTAIN PROBLEMS OF HINDU ASTRONOMY†

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ABSTRACT

Hindu methods of computing planetary positions of Sun, Moon etc. and calculating *Mandaphala* (Equation of Centre) and *Gatiphal* are some what lengthy. In this paper, we have suggested certain techniques of Modern Astronomy to compute planetary position, *Manadaphala* and *Gatiphala*. The methods are explained by taking examples from Hindu Astronomy.

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INTRODUCTION

It appears that *Bhāskarācārya* II (b.1114 A.D.) had notions of Differential Calculus, but certainly not in the modern language. The above idea, 500 years before Newton and Leibntz, appears in his famous treatise *Siddhānta Śiromaṇi* (SS). For the exact computation of the daily motion of planets Bhāskara II introduced the concept of *Tatakālika Gati* (instantaneous velocity) by dividing the day into a large number of small intervals and compared the positions of planets at the end of successive intervals. If y and y′ are the mean anomalies of the planets at the end of consecutive intervals, then according to him

$$H\sin y' - H\sin y \doteq (y' - y)\, H\cos y$$

(Here *Hsin* stands for Hindu-sine).

This result is equivalent to

$$\delta(\sin y) = \cos y\ \delta y$$

(see, [3, p.110], [4, p. 146-148] and [1, p.92]).

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Professor Srinivasiengar [op.cit.] observes that *Bhā skara* has gone deeper into Differential Calculus in *Siddhā nta Śiromaṇi* and suggests that the differential coefficient vanishes at an extreme value of the function. In the case of lunar and solar eclipses, he says that celestial latitude (β) of the moon plays an important role for the determination of *Sparsa-Sthiti-Khaṇda* (the time between the moment of first contact and the middle of eclipse, i.e. half the duration of an eclipse), *Mokṣa-Sthiti-Khaṇda* (the time between the middle of the eclipse and the moment of the last contact), *Sammilana-Marda-Khaṇda* (the time between commencement of total eclipse and the middle of the eclipse) and *Unmilana-Marda-Khaṇda* (the time between the middle of the total eclipse and the end of the eclipse). It is well-known that celestial latitude (β) of the moon varies from moment to moment. For the exact computation of celestial latitude (β) at a point of the orbit of the moon, *Bhāskara* introduced a method in the following two verses of the SS :

स्थित्यर्धनाडीगुणिता स्वभुक्तिः
षष्ट्या ६० हृता तद्रहितौ युतौ च ।
कृत्वेन्दुपा[illegible]सकृच्छराभ्यां
स्थित्यर्धमाद्यं स्फुटमन्तिमं च ॥
एवं विमदार्धफलोनयुक्त-
सपातचन्द्रोद्‌भवसायकाभ्याम् ।
पृथक् पृथक् पूर्ववदेवसिद्धे
स्फुट स्त आद्यन्त्यविमर्दखण्डे ॥

(The first verse gives the rectification of the times of *Sparsa-Sthiti-Khaṇda* and *Mokṣa-Sthiti-Khaṇda*, and the second verse gives the rectification of *Sammilana-Madra-Khaṇda* and *Unmilana-Marda-Khaṇda*).

However, when we follow the *Bhā skara's* method to calculate (β), we have to undergo somewhat lengthy calculations. Since the basic ideas of calculus are woven in the theory, we can apply modern calculus. For example to get the rectified value of β, let us differentiate the following with respect to β:

$$H\sin\beta = (H\sin\lambda\ H\sin i)/R$$

to get

$$H\cos\beta\delta\beta = (H\sin i\ H\cos\lambda\delta\lambda)/R$$

wherein i is the inclination of moon's orbit to ecliptic i.e., $i=5°8'40''$, $R=3438'$ and $(H\sin i)/R$ is a constant and λ = Moon's longitude - Node's longitude.

It may be mentioned that this formula gives in one stroke the rectified celestial latitude of the moon at the respectdive moments from which the

respective rectified time may be computed immediately. The calculus may also be applied in several other situations.
We illustrate the method by an example.

Example : Compute the *Sparsa-Sthiti-Khaṇda* and *Mokṣa-Sthiti-Khaṇda* of partial lunar eclipse on August 6, 1990 at Hardwar.

Step 1

First we get the rectified planetary positions after making necessary corrections of *Desāntara, Carajyā*, *Bhujāntara* and *Udayāntara* of the sun, the moon and the lunar node with the help of *ahargaṇa.*

The following table gives the desired planetary position on August 6, 1990 at Hardwar.

Rectified longitude of the sun (Rls)	$3^R\ 20^\circ\ 25'\ 49''$
Rectified longitude of the moon (Rls)	$9^R\ 15^\circ\ 35'\ 19''$
Longitude of the lunar node	$9^R\ 12^\circ\ 59'\ 58''$
Difference of the sun and the lunar node	$282^\circ 59'58''-110^\circ 25'45''=172^\circ 34'13''$
Distance of the sun from the lunar node	$180^\circ-172^\circ\ 34'13''=7^\circ\ 25'47'' < 9^\circ$
Rlm-Rls	$9^R 15^\circ 35'19''-3^R 20^\circ 25'45''=5^R 25^\circ 9'34''$ $=175^\circ\ 9'\ 34''$
Difference to cover 180°	$180^\circ-175^\circ\ 9'\ 34''=4^\circ 50'\ 26''=17426''$
Rectified motion (*Gati*) of the sun (s)	$58'\ 18''\ 10'''$
Rectified motion (*Gati*) of the moon	$765'\ 1''$
m-s	$765'\ 1''-58'\ 18''\ 10''' = 42463''$
Duration of *Purnima* beginning from the sun rise i.e. from 0548 hrs (IST)	$17426 \times 60/42463$ Ghatis 24.62284812 Ghatis $24.62284812 \times 24 \div 60$ hrs 9.849139246 hrs $9^h\ 50^m\ 56^s$

Evidently there is a lunar eclipse on Aug. 6, 1990.
Note that Rlm - Rls = 7° 25' 47" < 9°

Step II:

Rectification of the sun, the moon and the lunar node at the end of *Purṇimā*

The following table shows the rectified longitudes of the sun, the moon and the lunar node at the end of *Purṇimā* (On Aug. 6, 1990).

Sun's motion during 60 *Ghatis*	$57' 18'' 10''' = 57.302777'$
Sun's motion during *24.62284812 Ghatis*	$24.62284812 \times 57.302777/60 = 23'30''$
Rectified longitudeof the sun at the end of *Purṇimā*	$3^R 20^\circ 25' 45'' + 23'30'' = 3^R 20^\circ 49' 15''$
Moon's motion during *24.62284812 Ghatis*	$24.62284812 \times 765.01/60 = 5^\circ 13'56''$
Rectified longitude of the moon at the end of *Purṇimā*	$9^R 15^\circ 35' + 5^\circ 13'56'' = 9^R 20^\circ 49' 15''$
Mean daily motion of the lunar node	$0\text{-}3'\text{-}10''\text{-}48''\text{-}20''' = 3' 11'''$ approx.
Increment of the lunar node	$24.62284812 \times 3.180092592/60 = 1'18''$
Rectified longitude of the lunar node	$9^R 12^\circ 59' 58'' - 1' 18'' = 9^R 12^\circ 58' 40''$

ρ = Angular radius of the shadow cone $= (2m/15-5s/12)'/2$
$= (2 \times 765.01/15-5 \times 57.3027/12)'/12$
$= 39.0626'$

r = Angular radius of the moon $= 3m'/148 = 15.50'$

Moon's longitude with respect to the lunar node $= 9^R 20^\circ 49' 15'' - 9^R 12^\circ 58'40''$
$= 7^\circ 50' 35''$

Celestial latitude (β) of the moon = (Hsin λ × Hsin i)/3438
$= 3438 \sin 7^\circ 50'35'' \times 270' \div 3438 = 36.38''$

Now $\rho + r - \beta = 17.93'$

CONDITION FOR LUNAR ECLIPSE

Total lunar eclipse occurs when $\rho+r-\beta>2r$ and a partial lunar eclipse occurs when $\rho+r-\beta<2r$ since $\rho+r-\beta=17.39'<31'$. There is a partial lunar eclipse on Aug. 6, 1990.

Sparsā-Sthiti-Khaṇda = $[(\rho+r)^2-\beta^2]^{1/2}\times 60\div m\text{-}s$ *Ghatis*
(The first half duration of the eclipse) = *3.427754943 Ghatis*

Therefore the length of the lunar eclipse is twice this period, i.e. *6.85550986 Ghatis*. This value is further refined 2 hrs. and 44.5 minutes.

Step III

The Following *Bhāskara II* (cf. second verse above), we get the following:

$\lambda_1 = 7^\circ\ 6'\ 33'$ and $\lambda_2 = 8^\circ\ 34'\ 27''$

Wherein λ_1 (respectively λ_2) is the difference of the longitude of the moon and its node at the first (respectively last point of contact).

Also $\beta_1 = 33.372'$ and $\beta_2 = 39.90'$

where β_1 and β_2 are the rectified values of β at the beginning and end of the eclipse. These vlaues may be calculated in a manner analogeous to β. (Further rectification of β_2 gives the same value).

STEP IV

Refined-*Sparsa-Sthiti-Khaṇda* = $[(\rho+r)^2-\beta_1^2]^{1/2}\times 60\div m\text{-}s$
= *3.699716455 Ghatis*
= *1 hr 27 m 49 sec.*

Mokṣa-Sthiti-Khaṇda = $\left[(\rho+r)^2-\beta_2^2\right]^{1/2}\times 60\div m-s$
= *3.155232422 Ghatis*
= *1 hr 15 m 43 sec.*

Now we apply modern calculus to get the rectified value of celestial latitude of the moon, we use the calculations of step I-III. Differentiating the formula :

$$H\sin\beta = 270'\times H\sin\lambda \div 3438\ Kal\bar{a}$$

we get

$$\delta\beta = 270'\times \cos\lambda\,\delta\lambda \div 3438\cos\beta\ Kal\bar{a}$$

$$= 270' \cos 7^\circ 6' 53'' \times 1^\circ 27' 44'' \div 3438 \cos 36.36'\ Kal\bar{a}$$
$$= 6.48\ Kal\bar{a}$$

In fact this is the approximate average change in the celestial latitude of the moon.

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